

# EQUIVARIANT HOMOTOPY DIAGRAMS

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ABSTRACT. We discuss obstructions to extending equivariant homotopy commutative diagrams to more highly commutative ones and rectifications of these equivariant homotopy commutative diagrams. As an application we show that in some cases these obstructions vanish and the rectifications can be used to construct group actions on products of spheres.

## 1. INTRODUCTION

Let  $\mathcal{A}$  and  $\mathcal{M}$  be two categories and  $U: \mathcal{A} \rightarrow \mathcal{M}$  be a forgetful functor. We will say an object  $Y$  of  $\mathcal{A}$  is a free object if there exists an object  $X$  of  $\mathcal{M}$  and a morphism  $\eta: X \rightarrow U(Y)$  in  $\mathcal{M}$  such that  $(Y, \eta)$  is an initial object in the comma category  $X \downarrow U$ . For example when you consider the forgetful functor from the category of groups to the category of sets, free objects are free groups. Similarly one can define free modules. However in some categories it is notoriously hard to construct free objects. In this paper, we discuss one such category and a method for constructing free objects in that category.

Let  $G$  be a finite group,  $\mathcal{M}$  be a category, and  $\mathcal{G}_G$  denote the groupoid which has one object  $*_G$  with morphism set equals to the group  $G$ . A functor from  $\mathcal{G}_G$  to  $\mathcal{M}$  will be called a  $G$ -object of  $\mathcal{M}$ . Given  $X$  a  $G$ -object of  $\mathcal{M}$  we can obtain a functor  $\bar{X}$  from the opposite category of the poset of subgroups of  $G$  to  $\mathcal{M}$  by sending  $H$  to limit of  $X$  over  $\mathcal{G}_H$  if all these limits exist. From now on, we assume that  $\mathcal{M}$  is a category for which all the limits  $\lim_{\mathcal{G}_H} X$  exist.

Let  $\mathcal{A}$  denote the category with  $G$ -objects of  $\mathcal{M}$  as objects and natural transformations from  $\bar{X}$  to  $\bar{Y}$  as morphisms between  $X, Y$  two  $G$ -objects of  $\mathcal{M}$ . A free  $G$ -object is a free object for the forgetful functor  $U: \mathcal{A} \rightarrow \mathcal{M}$  which send  $X$  to  $X(*_G)$ . For example if  $\mathcal{M} = \mathbf{Top}$  the category of topological spaces and continuous functions then free  $G$ -objects are spaces with a free  $G$ -action. In this case we know that there is no free  $\mathbb{Z}/3$ -object  $Y$  with  $U(Y) = \mathbb{S}^2$ .

Let  $\mathcal{M}, \mathcal{A}$  be as above. If we further assume that  $\mathcal{M}$  is a homotopical category then we have a localization functor  $l$  from  $\mathcal{M}$  to its homotopy category  $\mathrm{Ho}(\mathcal{M})$ . For an object  $X$  of  $\mathcal{M}$ , we will still write  $X$  instead of  $l(X)$  which is an object of  $\mathrm{Ho}(\mathcal{M})$ . Considering the weak equivalences on  $\mathcal{M} = \mathbf{Top}$  defined by isomorphisms of homotopy groups, we have  $Y = EG \times \mathbb{S}^2$  as a free  $G$ -object with  $U(Y) \cong \mathbb{S}^2$  in  $\mathrm{Ho}(\mathcal{M})$ .

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Considering diffeomorphisms as weak equivalences on  $\mathcal{M} = \mathbf{Man}$  the category of smooth manifolds and smooth maps, the existence of free  $G$ -objects  $Y$  with  $U(Y) \cong \mathbb{S}^3$  in  $\mathrm{Ho}(\mathcal{M})$  are studied as a part of the generalized Poincare conjecture.

Here the case we are interested in is the case when  $\mathcal{M} = \mathbf{CW}$  the category of finite CW-complexes as objects and admissible cellular maps as morphisms. Admissible means open cells are sent homeomorphically onto open cells and such homeomorphisms between open cells commute with the identity map on the interior of the disk after possible compositions with their characteristic maps. Here free  $G$ -objects of  $\mathbf{CW}$  are free  $G$ -CW-complexes. Consider weak equivalences in  $\mathbf{CW}$  as the weak equivalences of the underlying topological spaces. Given a group  $G$  and a product of  $k$  spheres  $\mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_k}$ , we will try to find if there exist a free  $G$ -object  $X$  of  $\mathbf{CW}$  such that  $X \cong \mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_k}$  in  $\mathrm{Ho}(\mathbf{CW})$ .

We say a  $\mathcal{G}$ -object  $X: \mathcal{G} \rightarrow \mathcal{M}$  has isotropy in  $\mathcal{F}$  if for all  $\mathcal{H}$  subgroupoid of  $\mathcal{G}$  not in  $\mathcal{F}$  the limit of  $X$  over  $\mathcal{H}$  is an initial object. Notice that  $X$  is a free  $G$ -object if it has isotropy in the family which only contains the trivial subgroup of  $G$ . So after fixing a homotopy action the construction of free  $G$ -objects in  $\mathbf{Top}$  is the same as constructing objects with certain fixed point systems which is discussed in [8],[19], [13], and [4]. However there are obstructions to these constructions. Here we discuss a recursive method for handling these obstructions.

Swan [21] proved that there exists a free  $G$ -object of  $\mathbf{CW}$  if  $G$  does not contain a subgroup isomorphic to  $\mathbb{Z}/p \times \mathbb{Z}/p$  for any prime  $p$ . The converse of this result is proved by Smith [20]. Define the rank of a group  $G$  as the number  $k$  such that  $G$  contains a subgroup isomorphic to an elementary abelian group of rank  $k$  but no subgroup isomorphic to an elementary abelian group of rank  $k+1$ . It is conjectured (see [5]) that if the finite group  $G$  has rank less than or equal to  $k$ , there exist a finite  $G$ -CW-complex homotopy equivalent to a product of  $k$  spheres. So this conjecture is known to be true in case  $k=1$ . The conjecture is also proved for  $k=2$  when  $Qd(p)$  is not involved in the groups we consider (see [2], [15]).

Madsen-Thomas-Wall [18] proved that a finite group  $G$  acts freely on a sphere if (i)  $G$  has no subgroup isomorphic to the elementary abelian group  $\mathbb{Z}/p \times \mathbb{Z}/p$  for any prime number  $p$  and (ii) has no subgroup isomorphic to the dihedral group  $D_{2p}$  of order  $2p$  for any odd prime number  $p$ . It is also known (see [14], [1], [25]) that a finite  $p$ -group can act freely and smoothly on a product of two spheres if and only if the rank of the group is less than or equal to 3. Methods developed in these results were also used in [26] to show that for every finite group  $G$  there exists a finite free  $G$ -CW-complex homotopy equivalent to a product of spheres where the action is homologically trivial.

In these later results, the idea is to construct  $X_n \rightarrow \cdots X_1 \rightarrow X_0 = *$  a sequence of  $G$ -objects and  $\{\{e\}\} = \mathcal{F}_n \subseteq \cdots \subseteq \mathcal{F}_0$  a sequence of subgroup families of  $G$  so that  $X_i$  has isotropy in  $\mathcal{F}_i$  and  $X_i \cong X_{i-1} \times \mathbb{S}^{n_i}$  in the chosen homotopy category for some  $n_i \geq 1$ . In the construction of  $X_i$  from  $X_{i-1}$ , the starting point is a diagram in the homotopy category. However gluing these over  $X_{i-1}$  has some obstructions.

Here we discuss these obstructions. These obstructions are similar to the obstructions discussed in [7] and [12]. More precisely we replace homotopy diagrams with cubically enriched functors called homotopy commutative diagrams and discuss their rectifications which can be used to construct the cubically enriched functors in the next step of the recursive process.

In Sections 2, 3 and 4, we remind some basic definitions and facts about simplicial sets and cubical sets and set the notations we will be using. In Section 5, we introduce a bar construction which will be used to rectify homotopy commutative diagrams. In Section 6, we discuss properties of the homotopy coherent nerve that we obtain by using the bar construction of the previous section. In Section 7, we discuss homotopy commutative diagrams and maps between homotopy commutative diagrams. In Section 8, we discuss the obstructions to extending homotopy commutative diagrams to ones that are more highly commutative. In the last section, we give some applications.

## 2. SOME CLOSED SYMMETRIC MONOIDAL CATEGORIES

Let **Set** denote the category of sets, **Top** denote the category of compactly generated topological spaces, and **Cat** denote the category of small categories. These are all examples of closed symmetric monoidal categories with cartesian product.

**2.1. Simplicial sets.** Let  $[n]$  denote the category with objects  $\{0, 1, 2, \dots, n\}$  and exactly one morphism from  $i$  to  $j$  when  $i < j$ . Let  $\Delta$  be the finite ordinal number category, i.e., the full subcategory of **Cat** with objects  $[n]$  for  $n$  a non-negative integer. Let  $\mathcal{C}$  be a small category. Then simplicial objects of  $\mathcal{C}$  are functors from  $\Delta^{op}$  to  $\mathcal{C}$ . Simplicial objects of  $\mathcal{C}$  form a category denoted by  $\mathbf{s}\mathcal{C}$  with morphisms given by natural transformations. If  $\mathcal{D}$  is a small category and  $\mathcal{C}$  is an arbitrary category, the category of functors from  $\mathcal{D}$  to  $\mathcal{C}$  is denoted by  $\mathcal{C}^{\mathcal{D}}$ . Hence we have  $\mathbf{s}\mathcal{C} = \mathcal{C}^{\Delta^{op}}$ . For  $K$  a simplicial object of a category, we write  $K_n$  instead of  $K([n])$  and call it the  $n$ -simplices of  $K$ . For  $i$  in  $\{0, 1, 2, \dots, n\}$ , we define a functor  $d^i$  from  $[n-1]$  to  $[n]$  by

$$d^i(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases}$$

and define a functor  $s^i$  from  $[n+1]$  to  $[n]$  by

$$s^i(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i. \end{cases}$$

It is known that all morphisms of  $\Delta$  can be written as compositions of the above morphisms. For a simplicial object  $K$ , the map  $d_i: K_n \rightarrow K_{n-1}$  induced by  $d^i$  is called an  $i$ -th face map of  $K$ , and the map  $s_i: K_n \rightarrow K_{n+1}$  induced by  $s^i$  is called an  $i$ -th degeneracy map.

Objects of  $\mathbf{sSet}$  are called simplicial sets. There is a standard embedding of  $\Delta$  in  $\mathbf{sSet}$  which takes  $[n]$  to the standard  $n$ -simplex  $\Delta^n =: \Delta(-, [n]): \Delta^{op} \rightarrow \mathbf{Set}$ . Let  $\Delta_{\leq n}$  be the full subcategory of the finite ordinal number category on the objects  $\{[0], [1], \dots, [n]\}$ . The functor induced from the inclusion  $\Delta_{\leq n} \hookrightarrow \Delta$  is called the  $n$ -truncation functor  $\tau_n: \mathbf{Set}^{\Delta_{\leq n}^{op}} \rightarrow \mathbf{sSet}$ . This functor has a left adjoint and composing two gives the  $n$ -th skeleton functor  $sk_n: \mathbf{sSet} \rightarrow \mathbf{sSet}$ . Therefore, given a simplicial set  $K$ ,  $sk_n K$  is the subsimplicial set of  $K$  generated by  $K_0, \dots, K_n$  under the degeneracies. If  $K$  and  $L$  are two simplicial sets, their simplicial tensor  $K \otimes L$  in  $\mathbf{sSet}$  is defined:

$$(K \otimes L)_n := K_n \times L_n.$$

This defines a symmetric monoidal structure on simplicial sets.

**2.2. Cubical sets.** Let  $n$  be a non-negative integer. We will write  $\mathcal{I}^n$  to denote the set of functions from  $\{1, 2, \dots, n\}$  to  $\{0, 1\}$  for  $n \geq 0$ . Here  $\mathcal{I}^0$  contains a single function namely the empty function. For  $i$  in  $\{1, 2, \dots, n\}$  and  $\varepsilon$  in  $\{0, 1\}$ , define a function  $d^{(i, \varepsilon)}$  from  $\mathcal{I}^{n-1}$  to  $\mathcal{I}^n$  by

$$d^{(i, \varepsilon)}(f)(j) = \begin{cases} f(j) & \text{if } j < i \\ \varepsilon & \text{if } j = i \\ f(j-1) & \text{if } j > i \end{cases}$$

for  $f$  in  $\mathcal{I}^{n-1}$  and define a function  $s^i$  from  $\mathcal{I}^n$  to  $\mathcal{I}^{n-1}$  by

$$s^i(f)(j) = \begin{cases} f(j) & \text{if } j < i \\ f(j+1) & \text{if } j \geq i \end{cases}$$

for  $f$  in  $\mathcal{I}^n$  and define a function  $c^i$  from  $\mathcal{I}^{n+1}$  to  $\mathcal{I}^n$  by

$$c^i(f)(j) = \begin{cases} f(j) & \text{if } j < i \\ \max(f(j), f(j+1)) & \text{if } j = i \\ f(j+1) & \text{if } j > i \end{cases}$$

for  $f$  in  $\mathcal{I}^{n+1}$ . Let  $\square_c$  denote the Box category with connections, whose objects are  $\{\mathcal{I}^n\}_{n=0}^\infty$  and the morphisms of  $\square_c$  are generated by the functions  $d^{(i, \varepsilon)}$ ,  $s^i$  and  $c^i$  where  $n \geq 0$  and  $1 \leq i \leq n$ . Let  $\mathcal{C}$  be a category, a functor  $K: \square_c^{op} \rightarrow \mathcal{C}$  is called a cubical object of  $\mathcal{C}$ . We write  $K_n$  instead of  $K(\mathcal{I}^n)$  and call it the  $n$ -cubes of  $K$ . The two maps  $d_{(i, \varepsilon)}: K_{n-1} \rightarrow K_n$  induced by  $d^{(i, \varepsilon)}$  is called an  $i$ -th face map of  $K$ , the map  $s_i: K_n \rightarrow K_{n-1}$  induced by  $s^i$  is called an  $i$ -th degeneracy map, and the map  $c_i: K_{n+1} \rightarrow K_n$  induced by  $c^i$  is called an  $i$ -th connection map. The collection of all cubical objects of a category  $\mathcal{C}$  form a category which we denote by  $\mathbf{cC}$ , in other words,  $\mathbf{cC}$  denotes the functor category from  $\square_c^{op}$  to  $\mathcal{C}$ . Objects of  $\mathbf{cSet}$  are called cubical sets. There is a standard embedding of  $\square_c$  in  $\mathbf{cSet}$  called the standard  $n$ -cube  $I^n$  which is the cubical set  $\square_c(-, \mathcal{I}^n): \square_c^{op} \rightarrow \mathbf{Set}$ . For a cubical set  $K$ , the subcubical set of  $K$  generated by  $K_0, \dots, K_n$  under the degeneracies is called the

$n$ -th cubical skeleton of  $K$ , written  $\text{sk}_n^c K$ . As in the simplicial sets, one could also see the  $n$ -skeleton functor  $\text{sk}_n^c: \mathbf{cSet} \rightarrow \mathbf{cSet}$  as a composition of the  $n$ -truncation functor  $\tau_n$  on cubical sets with its left adjoint. If  $K$  and  $L$  are two cubical sets, their cubical tensor  $K \otimes L$  in  $\mathbf{cSet}$  is defined:

$$K \otimes L := \text{colim}_{I^j \rightarrow K, I^k \rightarrow L} I^{j+k}.$$

This defines a symmetric monoidal structure on cubical sets.

There are adjoint functors

$$\mathbf{cSet} \rightleftarrows_S^T \mathbf{sSet}$$

relating the cubical sets and the simplicial sets. The singular functor  $S$  associates to every simplicial set  $X$ , a singular cubical set  $S(X)$  with  $n$ -cells  $S(X)_n = \mathbf{sSet}((\Delta^1)^n, X)$ . The triangulation functor  $T$  is defined by  $TK = \text{colim}_{I^n \rightarrow K} (\Delta^1)^n$ . These functors induce an equivalence of homotopy categories.

### 3. SOME ENRICHED CATEGORIES

Let  $\mathcal{V}$  be a bicomplete closed symmetric monoidal category with unit object  $E$  and product  $\otimes$ . Let  $[B, C]$  denote the internal hom-object from  $B$  to  $C$ . The product and the internal hom-objects satisfy an adjunction

$$\mathcal{V}(A \otimes B, C) \cong \mathcal{V}(A, [B, C]).$$

A category  $\mathcal{M}$  enriched over  $\mathcal{V}$  means that  $\mathcal{M}(M, M')$  is an object of  $\mathcal{V}$  for every  $M, M'$  object of  $\mathcal{M}$  and the composition of  $\mathcal{M}$  is given by a morphism of  $\mathcal{V}$

$$\mathcal{M}(M', M'') \otimes \mathcal{M}(M, M') \rightarrow \mathcal{M}(M, M''),$$

that satisfies the associativity and unit axioms.

Note that each  $\mathcal{V}$ -category  $\mathcal{M}$  gives rise to an ordinary category  $\mathcal{M}_0$  which has the same objects and  $\mathcal{M}_0(M, M') = \mathcal{V}(E, \mathcal{M}(M, M'))$ .

A  $\mathcal{V}$ -functor  $F: \mathcal{M} \rightarrow \mathcal{N}$  between  $\mathcal{V}$ -categories  $\mathcal{M}$  and  $\mathcal{N}$  assign an object  $F(M)$  of  $\mathcal{N}$  to each object  $M$  of  $\mathcal{M}$  and a morphism

$$\mathcal{M}(M, M') \rightarrow \mathcal{N}(F(M), F(M'))$$

in  $\mathcal{V}$  that satisfy the functoriality diagrams. One can similarly define a  $\mathcal{V}$ -adjoint pair and a  $\mathcal{V}$ -natural transformation. In particular, these notions give rise to an underlying ordinary functor  $F_0: \mathcal{M} \rightarrow \mathcal{N}$ , underlying ordinary adjoint pairs and natural transformation.

**3.1. Cofibrant replacement of simplicially enriched categories.** One way of assigning a simplicial category to a small category is the standard resolution construction of [9] and [10] (see also [22]). Given a small category  $\mathcal{C}$ , the free category on  $\mathcal{C}$  is the free category  $\mathcal{FC}$ , where  $\text{obj}(\mathcal{FC}) = \text{obj}(\mathcal{C})$ , and a morphism in  $\mathcal{FC}$  from  $c$  to  $d$  is a word of composable non-identity morphisms  $(f_n, \dots, f_1)$  such that the domain of  $f_1$  is  $c$  and the range of  $f_n$  is  $d$ . Here the composition is given by the concatenation

of words and the identity morphism on  $c$  is the empty word  $()$ . For simplicity, we write  $\mathcal{F}f := (f) \in \mathcal{FC}(c, d)$  for a morphism  $f \in \mathcal{C}(c, d)$ .

For every functor  $F: \mathcal{C} \rightarrow \mathcal{D}$ , there is an induced functor  $\mathcal{F}F: \mathcal{FC} \rightarrow \mathcal{FD}$  which sends  $\mathcal{F}f$  to  $\mathcal{F}(F(f))$ . This makes  $\mathcal{F}$  a functor from small categories to small categories. The functor  $\mathcal{F}$  has a co-monad structure with the co-unit functor  $U_{\mathcal{C}}: \mathcal{FC} \rightarrow \mathcal{C}$  which is the identity on objects and sends  $\mathcal{F}f$  to  $f$ . The co-multiplication functor  $\Delta_{\mathcal{C}}: \mathcal{FC} \rightarrow \mathcal{F}^2\mathcal{C}$  is defined by  $\Delta_{\mathcal{C}}(\mathcal{F}f) = \mathcal{F}^2f$ . Clearly these functors satisfy the comonad identities. Therefore one can construct a simplicial category (with discrete object set)  $\mathcal{F}_{\bullet}\mathcal{C}$  by letting  $\mathcal{F}_n\mathcal{C} = \mathcal{F}^{n+1}\mathcal{C}$ . Here the face and the degeneracy maps are identity on objects and on morphisms they are given by

$$\begin{aligned} d_i &= \mathcal{F}^i(U_{\mathcal{F}^{n+1-i}\mathcal{C}}): \mathcal{F}^{n+1}\mathcal{C}(c, d) \rightarrow \mathcal{F}^n\mathcal{C}(c, d) \\ s_i &= \mathcal{F}^i(\Delta_{\mathcal{F}^{n-i}\mathcal{C}}): \mathcal{F}^{n+1}\mathcal{C}(c, d) \rightarrow \mathcal{F}^{n+2}\mathcal{C}(c, d). \end{aligned}$$

The category  $\mathcal{F}_{\bullet}\mathcal{C}$  is also a simplicially enriched category.

**3.2. Cofibrant replacement of cubically enriched categories.** The  $W$ -construction on a small category  $\mathcal{C}$  is the **cSet**-category  $W\mathcal{C}$  where  $\text{Obj}(W\mathcal{C}) = \text{Obj}(\mathcal{C})$  and for every  $a, b$  objects of  $\mathcal{C}$ , the cubical mapping complex  $W\mathcal{C}(a, b)$  constructed as follows:

$$W\mathcal{C}(a, b) = \left( \coprod_{\substack{\sigma: [n+1] \rightarrow \mathcal{C}, \\ \sigma(0)=a, \sigma(n+1)=b}} I_{\sigma}^n \right) / \sim$$

where the equivalence relation is given by

$$I_{\sigma d^{n+1-i}}^{n-1} \sim d^{i,0}(I_{\sigma}^n) \text{ and } I_{\sigma s^{n+1-i}}^{n+1} \sim \begin{cases} s_i(I_{\sigma}^n) & \text{if } i = 1, n+1 \\ c_i(I_{\sigma}^n) & \text{otherwise.} \end{cases}$$

The cubical composition

$$W\mathcal{C}(a_0, a_i) \otimes W\mathcal{C}(a_i, a_{n+1}) \rightarrow W\mathcal{C}(a_0, a_{n+1}) = W\mathcal{C}(a, b)$$

sends the  $(n-1)$ -cube  $I_{\beta}^{i-1} \otimes I_{\sigma}^{n-i}$  to  $d^{(i,1)}I_{\beta \cdot \sigma}^n$  where  $(\beta \cdot \sigma)(j \rightarrow j+1) = \sigma(j \rightarrow j+1)$  if  $j \leq n-i$  and  $(\beta \cdot \sigma)(j \rightarrow j+1) = \beta((j+i-n-1) \rightarrow (j+i-n))$ , otherwise.

Moreover for every functor  $f: \mathcal{C} \rightarrow \mathcal{D}$ , the  $W$ -construction induces a **cSet**-functor  $Wf: W\mathcal{C} \rightarrow W\mathcal{D}$  which is given on objects by  $f$  and on morphisms by sending an  $n$ -cube  $I_{\sigma}^n$  to  $I_{f \circ \sigma}^n$ . This makes  $W$  a functor from the category of small categories to the category of **cSet**-categories. For every small category  $\mathcal{C}$ , the simplicially enriched categories  $TW\mathcal{C}$  and  $\mathcal{F}_{\bullet}\mathcal{C}$  are naturally equivalent (see Lemma 3.6 in [3]).

#### 4. SOME TENSORED CATEGORIES

We say that a  $\mathcal{V}$ -category  $\mathcal{M}$  is tensored if for every object  $K$  of  $\mathcal{V}$  and  $M$  of  $\mathcal{M}$ , there is an object  $K \odot M$  of  $\mathcal{M}$  and a  $\mathcal{V}$ -natural isomorphism:

$$[K, \mathcal{M}(M, M')] \cong \mathcal{M}(K \odot M, M').$$

**Definition 4.1.** Let  $\mathcal{M}$  be a tensored  $\mathcal{V}$ -category,  $\mathcal{D}$  be a small  $\mathcal{V}$ -category, and  $G: \mathcal{D}^{op} \rightarrow \mathcal{V}$  and  $F: \mathcal{D} \rightarrow \mathcal{M}$  be  $\mathcal{V}$ -functors. Then their tensor product is defined to be the following coequalizer:

$$(1) \quad G \odot_{\mathcal{D}} F = \text{coeq} \left( \coprod_{d, d'} (Gd' \otimes \mathcal{D}(d, d')) \odot Fd \rightrightarrows \coprod_d Gd \odot Fd \right).$$

**4.1. Geometric realization.** Let  $\mathcal{M}$  be a tensored  $\mathcal{V}$ -category,  $\mathcal{D}$  be a small  $\mathcal{V}$ -category, and  $R: \mathcal{D} \rightarrow \mathcal{M}$  be a  $\mathcal{V}$ -functor. We can then define the geometric realization with respect to  $R$  of any functor  $X: \mathcal{D}^{op} \rightarrow \mathcal{V}$  to be

$$|X|_R = X \odot_{\mathcal{D}} R.$$

**Proposition 4.2.** Let  $f: D_1 \rightarrow D_2$  be a fibration and  $R: D_2 \rightarrow \mathcal{M}$ . Then for every functor  $X: D_1^{op} \rightarrow \mathcal{V}$ , there is a homeomorphism between  $|X|_{f^*(R)}$  and  $|\tilde{X}|_R$  where  $\tilde{X}$  is the left Kan extension of  $X$  along  $f^{op}$ .

*Proof.* Since  $\tilde{X}(c) = \text{colim}_{z \in f^{-1}(c)} X$  for every  $c \in D_2$ , the following diagram commutes

$$\begin{array}{ccc} \coprod_{x, y \in D_1} X(y) \otimes \text{Mor}_{D_1}(x, y) \odot R(f(x)) & \rightrightarrows & \coprod_{z \in D_1} X(z) \odot R(f(z)) \\ \downarrow & & \downarrow \\ \coprod_{a, b \in D_2} \tilde{X}(b) \otimes \text{Mor}_{D_1}(a, b) \odot R(a) & \rightrightarrows & \coprod_{c \in D_2} \tilde{X}(c) \odot R(c). \end{array}$$

Therefore  $|\tilde{X}|_R$  is a co-cone for the first row. In particular, we have

$$\tilde{X}(c) \odot R(c) = \text{colim}_{z \in f^{-1}(c)} X \odot R(c)$$

which makes  $|\tilde{X}|_R$  the co-equalizer. □

**4.2. Topological spaces.** The category **Top** can be given the structure of a **sSet**-category by applying the total singular complex functor to each hom-space. We will denote the resulting **sSet**-category by **Top<sub>sSet</sub>**. The category **Top<sub>sSet</sub>** is tensored with  $K \odot_{\mathbf{sSet}} X = |K| \times X$ .

Similarly we denote by **Top<sub>cSet</sub>** the **cSet**-category obtained by setting

$$\mathbf{Top}_{\mathbf{cSet}}(X, Y)_n = \mathbf{Top}(X \times I^n, Y).$$

The category **Top<sub>cSet</sub>** is tensored with  $K \odot_{\mathbf{cSet}} X := |K| \times X$ . Therefore for every cubical set  $A$  and a space  $X \in \mathbf{Top}$ , we have the following equivalence

$$TA \odot_{\mathbf{sSet}} X \simeq A \odot_{\mathbf{cSet}} X.$$

## 5. EQUIVARIANT DOUBLE BAR CONSTRUCTION

**5.1. Grothendieck constructions.** Let  $\mathbf{Cat}$  denote the category of small categories and functors. Given a functor  $\gamma: \mathcal{C} \rightarrow \mathbf{Cat}$  we define two categories called Grothendieck constructions of  $\gamma$ .

The category  $\int_{\mathcal{C}} \gamma$  is a category with objects

$$\{(c, x) \mid c \text{ is an object of } \mathcal{C} \text{ and } x \text{ is an object of } \gamma(c)\},$$

with morphisms from  $(c_1, x_1)$  to  $(c_2, x_2)$  is

$$\{(g, f) \mid g: c_1 \rightarrow c_2 \text{ is a morphism in } \mathcal{C} \text{ and } f: \gamma(g)(x_1) \rightarrow x_2 \text{ is a morphism in } \gamma(c_2)\},$$

with composition

$$(g_2, f_2) \circ (g_1, f_1) = (g_2 \circ g_1, f_2 \circ \gamma(g_1)(f_1)).$$

The category  $\int_{\mathcal{C}}^{op} \gamma$  is a category with objects

$$\{(c, x) \mid c \text{ is an object of } \mathcal{C} \text{ and } x \text{ is an object of } \gamma(c)\},$$

with morphisms from  $(c_1, x_1)$  to  $(c_2, x_2)$  is

$$\{(g, f) \mid g: c_1 \rightarrow c_2 \text{ is a morphism in } \mathcal{C} \text{ and } f: x_2 \rightarrow \gamma(g)(x_1) \text{ is a morphism in } \gamma(c_2)\},$$

with composition

$$(g_2, f_2) \circ (g_1, f_1) = (g_2 \circ g_1, \gamma(g_1)(f_1) \circ f_2).$$

Let  $\mathcal{V}\text{-Cat}$  denote the category of small  $\mathcal{V}$ -categories and  $\mathcal{V}$ -functors. From now on we assume that  $\mathcal{V}$  is closed under coproducts. Hence for any functor  $\gamma: \mathcal{C} \rightarrow \mathcal{V}\text{-Cat}$  the Grothendieck constructions  $\int_{\mathcal{C}}^{op} \gamma$  and  $\int_{\mathcal{C}} \gamma$  can be considered as a  $\mathcal{V}$ -category.

**5.2. Definition of double bar construction.** We now introduce the enriched two-sided bar construction. If  $\mathcal{M}$  is a tensored  $\mathcal{V}$ -category,  $\mathcal{C}$  is a small category,  $\gamma: \mathcal{C} \rightarrow \mathcal{V}\text{-Cat}$  is a functor and  $G: \int_{\mathcal{C}}^{op} \gamma \rightarrow \mathcal{V}$  and  $F: \int_{\mathcal{C}} \gamma \rightarrow \mathcal{M}$  are  $\mathcal{V}$ -functors then the two-sided simplicial bar construction is a functor  $B_{\bullet}(G, \gamma, F): \mathcal{C} \rightarrow s\mathcal{M}$  is defined as follows. For  $c$  an object of  $\mathcal{C}$ , the  $n$ -simplices  $B_n(G, \gamma, F)(c)$  of the simplicial object  $B_{\bullet}(G, \gamma, F)(c)$  is given by

$$\coprod_{\substack{\alpha_0, \alpha_1, \dots, \alpha_n \\ \text{objects of } \gamma(c)}} (G(c, \alpha_n) \otimes \gamma(c)(\alpha_{n-1}, \alpha_n) \otimes \dots \otimes \gamma(c)(\alpha_0, \alpha_1)) \odot F(c, \alpha_0)$$

and the face and the degeneracy maps are defined by using composition in  $\gamma(c)$ , the evaluation maps  $\gamma(c)(\alpha_0, \alpha_1) \odot F(c, \alpha_0) \rightarrow F(c, \alpha_1)$  and  $\gamma(c)(\alpha_{n-1}, \alpha_n) \otimes G(c, \alpha_n) \rightarrow G(c, \alpha_{n-1})$ , and the insertion of identities  $E \rightarrow \gamma(c)(\alpha, \alpha')$ . For a morphism  $f: c \rightarrow d$  in the category  $\mathcal{C}$  a morphism from  $B_n(G, \gamma, F)(c)$  to  $B_n(G, \gamma, F)(d)$  is defined by considering the morphisms  $G(f, 1_{\gamma(f)(\alpha_n)}): G(c, \alpha_n) \rightarrow G(d, \gamma(f)(\alpha_n))$ ,  $\gamma(f): \gamma(c)(\alpha, \alpha') \rightarrow \gamma(d)(\gamma(f)(\alpha), \gamma(f)(\alpha'))$ , and  $F(f, 1_{\gamma(f)(\alpha_0)}): F(c, \alpha_0) \rightarrow F(d, \gamma(f)(\alpha_0))$ .



The notion of double bar construction for enriched categories was first introduced by Shulman [23] and this construction can be considered as an equivariant version of Shulman's double bar construction.

**5.3. Properties of double bar construction.** Let  $\mathcal{M}$  be a tensored  $\mathcal{V}$ -category,  $\mathcal{C}$  be a small category,  $\gamma_0, \gamma_1: \mathcal{C} \rightarrow \mathcal{V}\text{-Cat}$  be functors, and  $G: \int_{\mathcal{C}}^{op} \gamma_1 \rightarrow \mathcal{V}$  and  $F: \int_{\mathcal{C}} \gamma_1 \rightarrow \mathcal{M}$  be  $\mathcal{V}$ -functors. Then every natural transformation from  $N: \gamma_0 \rightarrow \gamma_1$  induces a  $\mathcal{V}$ -functor  $N^*F: \int_{\mathcal{C}} \gamma_0 \rightarrow \mathcal{M}$  which is defined on objects by  $N^*F(c, x) = F(c, N_c(x))$  for every  $c \in \mathcal{C}$  and  $x \in \gamma_0(c)$ . For a morphism  $(g, f): (c_1, x_1) \rightarrow (c_2, x_2)$  in  $\int_{\mathcal{C}} \gamma_0$ , we define  $N^*F(g, f) = F(g, N_{c_2}(f))$ . Similarly, one can define a  $\mathcal{V}$ -functor  $N^*G: \int_{\mathcal{C}}^{op} \gamma_0 \rightarrow \mathcal{V}$ . Hence  $N$  induces a natural transformation  $N^*$  from  $B_{\bullet}(N^*G, \gamma_0, N^*F)$  to  $B_{\bullet}(G, \gamma_1, F)$ .

Let  $\gamma: \mathcal{C} \rightarrow \mathcal{V}\text{-Cat}$  be a functor, and  $G: \int_{\mathcal{C}}^{op} \gamma \rightarrow \mathcal{V}$  and  $F: \int_{\mathcal{C}} \gamma \rightarrow \mathcal{M}$  be  $\mathcal{V}$ -functors. Then every functor  $p: \mathcal{D} \rightarrow \mathcal{C}$  from a small category  $\mathcal{D}$  induces a natural transformation  $p^*$  from  $B_{\bullet}(p^*G, \gamma \circ p, p^*F)$  to  $B_{\bullet}(G, \gamma, F)$  where  $p^*G: \int_{\mathcal{D}}^{op} \gamma \circ p \rightarrow \mathcal{V}$  and  $p^*F: \int_{\mathcal{D}} \gamma \circ p \rightarrow \mathcal{M}$  are induced  $\mathcal{V}$ -functors.

**5.4. Opfibrations.** Let  $\mathcal{G}$  be a groupoid and  $\gamma: \mathcal{G} \rightarrow \mathbf{Cat}$  be a functor. We define a functor

$$\bar{d}\gamma: \mathcal{G} \rightarrow \mathbf{Cat}$$

on each object  $a$  of  $\mathcal{G}$  by letting  $\bar{d}\gamma(a)$  be the full subcategory of the simplex category  $\Delta \downarrow \gamma(a)$  with object set consisting of  $\sigma: [n] \rightarrow \gamma(a)$  where  $n = 1$  or for each  $i$  the morphism  $\sigma(i \rightarrow i+1)$  is not identity. Assume that  $\bar{I}\gamma$  denotes the Grothendieck construction

$$\bar{I}\gamma = \int_{\mathcal{G}} \bar{d}\gamma.$$

Let  $p_{\gamma}$  denote the natural projection

$$p_{\gamma}: \bar{I}\gamma \rightarrow \mathcal{G}.$$

Note that  $p_{\gamma}$  is an opfibration. Now define

$$\bar{f}\gamma: \bar{I}\gamma \rightarrow \mathbf{Cat}$$

by  $\bar{f}\gamma(a, \sigma) = [n]$  when  $\sigma: [n] \rightarrow \gamma(a)$ . Note that we can consider  $\gamma$  as the left Kan extension  $\text{Lan}_{p_{\gamma}}(\bar{f}\gamma)$  with the universal natural transformation  $\epsilon: \bar{f}\gamma \rightarrow \gamma \circ p_{\gamma}$  given by  $\epsilon_{(a, \sigma)} = \sigma$ .

**Theorem 5.1.** *Let  $G: \int_{\mathcal{G}}^{op} W\gamma \rightarrow \mathbf{cSet}$  and  $F: \int_{\mathcal{G}} W\gamma \rightarrow \mathcal{M}$  be morphisms in  $\mathbf{cSet}\text{-Cat}$ . Then we have*

$$B(G, W\gamma, F) = \text{Lan}_p B(\epsilon^* p_{\gamma}^* G, W\bar{f}\gamma, \epsilon^* p_{\gamma}^* F).$$

*Proof.* For every  $a \in \mathcal{G}$  and  $\sigma: [n] \rightarrow \gamma(a)$ , define

$$\eta_{a,\sigma}^m: (B(\epsilon^* p_\gamma^* G, W\bar{f}\gamma, \epsilon^* p_\gamma^* F)(a, \sigma))_m \rightarrow (B(G, W\gamma, F)(a))_m$$

by sending the summand  $(G(a, W(\sigma)(\alpha_m)) \otimes W([n])(\alpha_{m-1}, \alpha_m) \cdots \otimes W([n])(\alpha_0, \alpha_1)) \odot F(a, W(\sigma)(\alpha_0))$  corresponding to  $\alpha_0, \dots, \alpha_m$  in  $W([n])$  to a summand corresponding to  $W(\sigma)(\alpha_0), \dots, W(\sigma)(\alpha_m)$  in  $W(\gamma(a))$  by a map induced from the functor  $W(\sigma): W([n]) \rightarrow W(\gamma(a))$ . For a map  $(g, f): (a, \sigma: [n] \rightarrow \gamma(a)) \rightarrow (b, \beta: [m] \rightarrow \gamma(b))$ , we have  $\beta f = \gamma(g)\sigma$  and hence  $W(\beta)W(f) = W(\gamma(g))W(\sigma)$ . This makes  $\eta = (\eta_{a,\sigma})$  a natural transformation.

For a given functor  $M: \mathcal{G} \rightarrow \mathbf{cSet-Cat}$  and a natural transformation

$$\alpha: B(\epsilon^* p_\gamma^* G, W\bar{f}\gamma, \epsilon^* p_\gamma^* F) \rightarrow M \circ p,$$

we define  $\mu: B(G, W\gamma, F) \rightarrow M$  as follows. For every  $a \in \mathcal{G}$ , define

$$\mu_a^1: B(G, W\gamma, F)(a)_1 \rightarrow M(a)_1$$

by sending  $(I_a^j \otimes I_\beta^k) \odot I_a^l$  in  $(G(a, x_1) \otimes W(\gamma(a)(x_0, x_1)) \odot F(a, x_0))$  to  $\alpha_{a,\beta}(I_a^j \otimes I_{\iota_{k+1}}^k) \odot I_a^l$  where  $\iota_{k+1}: [k+1] \rightarrow [k+1]$  is the identity functor. For  $s > 1$ , define  $\mu_a^s$  by using the cubical decomposition defined in Section 3.2. Since  $\alpha$  is a natural transformation, we have  $M(f)\alpha_{(a,\beta)} = \alpha_{(b,\gamma(f)(\beta))}B(f, \text{id}_{\gamma(\beta)})$  for every map  $f: a \rightarrow b$  and  $\beta: [k+1] \rightarrow \gamma(a)$ . Therefore  $M(g)(\mu_a^1((I_a^j \otimes I_\beta^k) \odot I_a^l)) = \mu_b^1 B(f, \text{id}_{\gamma(\beta)})((I_a^j \otimes I_\beta^k) \odot I_a^l)$  and hence  $\mu$  is a natural transformation.

Note that for every  $(a, \sigma: [n] \rightarrow \gamma(a)) \in \bar{f}\gamma$  and  $g: [k+1] \rightarrow [n]$ , the following diagram commutes

$$\begin{array}{ccc} B(\epsilon^* p_\gamma^* G, W\bar{f}\gamma, \epsilon^* p_\gamma^* F)(a, \sigma \circ g) & \xrightarrow{B(\epsilon^* p_\gamma^* G, W\bar{f}\gamma, \epsilon^* p_\gamma^* F)(\text{id}_a, f)} & B(\epsilon^* p_\gamma^* G, W\bar{f}\gamma, \epsilon^* p_\gamma^* F)(a, \sigma) \\ \alpha_{(a, \sigma \circ g)} \downarrow & & \alpha_{(a, \sigma)} \downarrow \\ M(a) & \xlongequal{\quad} & M(a). \end{array}$$

Therefore we have  $\eta(\mu p) = \alpha$  as desired. Clearly  $\mu$  is unique.  $\square$

Note that the above theorem still holds if we replace the functor  $W$  with the functor  $\text{sk}_n^c W: \mathbf{Cat} \rightarrow \mathbf{cSet-Cat}$  or the functor  $T \text{sk}_n^c W: \mathbf{Cat} \rightarrow \mathbf{sSet-Cat}$ . Here the skeleton  $\text{sk}_n^c \mathcal{C}$  of a  $\mathbf{cSet}$ -category  $\mathcal{C}$  is generated by the tensor products of the cubes of dimension less than equal to  $n$ .

## 6. CLASSIFYING SPACES

**6.1. Category  $[n]$ .** The morphisms of the free simplicial category of  $[n]$  is given as follows (see [9])

$$\mathcal{F}_\bullet[n](i, j) = \begin{cases} \Delta[1]^{j-i-1}, & i \leq j; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Define

$$B_i^n = \coprod_{r=(r_0, \dots, r_i) \in R_i^n} |\mathcal{F}_\bullet[n](r_{i-1}, r_i)| \times \cdots \times |\mathcal{F}_\bullet[n](r_0, r_1)|$$

where  $R_i^n = \{r = (r_0, \dots, r_i) \mid 0 \leq r_0 < r_1 < \cdots < r_i \leq n\}$ . Then the standard geometric realization of the double bar construction  $B(*, \mathcal{F}_\bullet[n], *)$  is given by

$$|B(*, \mathcal{F}_\bullet[n], *)| = \left( \prod_{i \geq 0} B_i^n \times |\Delta^i| \right) / \sim$$

where the equivalence relation has  $(d_i x, t) = (x, d^i t)$  for every  $(x, t) \in B_i^n \times |\Delta^{i-1}|$ . From now on, we write  $|C^n(r)| := |\mathcal{F}_\bullet[n](r_{i-1}, r_i)| \times \cdots \times |\mathcal{F}_\bullet[n](r_0, r_1)|$  for short. We also denote an element  $(t_1^{(i)}, \dots, t_{r_i - r_{i-1} - 1}^{(i)}; \dots; t_1^{(1)}, \dots, t_{r_1 - r_0 - 1}^{(1)})$  of  $|C^n(r)|$  by  $t(r)$  and an element  $(t_0, \dots, t_i) \in |\Delta^i|$  by  $t$ .

Now consider  $B(*, \mathcal{F}_\bullet, *)$  as a functor from  $\tilde{\Delta}$  to **sSet** where  $\tilde{\Delta}$  is the subcategory of  $\Delta$  which has the same object set as  $\Delta$  and morphisms in  $\tilde{\Delta}$  are generated by the injections  $[k] \hookrightarrow [m]$  and the surjection  $[1] \rightarrow [0]$ .

**Theorem 6.1.** *There is a natural homeomorphism  $h: |B(*, \mathcal{F}_\bullet, *)| \rightarrow \Delta^\bullet$  where the functor  $\Delta^\bullet: \tilde{\Delta} \rightarrow \mathbf{Top}$  is given by  $\Delta^\bullet([n]) = |\Delta^n|$ .*

In order to define the map  $h_n := h([n])$ , we first introduce some notation. Let  $A_r^n$  be the subset of  $|\Delta^n|$  consisting of all  $a = (a_0, \dots, a_n)$  satisfying the following conditions

- (1)  $a_m = 0$  when  $m > r_i$  or  $m < r_0$ ,
- (2)  $a_u \leq \min(a_{r_k}, a_{r_{k+1}})$  when  $r_k < u < r_{k+1}$ ,
- (3)  $\min(a_{r_p}, a_{r_s}) \leq a_{r_{p+1}}$  when  $p < s$ ,

for each  $r = (r_0, \dots, r_i) \in R_i^n$  and  $1 \leq i \leq n$ . Now define  $h_n(r): |C_n(r)| \times |\Delta^i| \rightarrow A_r^n$  by

$$h_n(r)(t(r), t) = \frac{1}{\sum_{j=0}^n a_j} (a_0, \dots, a_n)$$

where

$$a_u = \begin{cases} t_{u-r_k}^{k+1} (\max_{p \leq k < s} \{\min(t_p, t_s)\}), & r_k < u < r_{k+1}, \\ 0, & u < r_0 \text{ or } u > r_i, \\ \max_{p < k < s} \{t_k, \min(t_p, t_s)\}, & u = r_k, \end{cases}$$

for every  $(t(r), t) \in |C_n(r)| \times |\Delta^i|$ . It easily follows that  $(a_0, \dots, a_n) \in A_r^n$ . Note that

$$d_j(t(r)) = (s_1^{(i-1)}, \dots, s_{d_j r_i - d_j r_{i-1} - 1}^{(i-1)}; \dots; s_1^{(1)}, \dots, s_{d_j r_1 - d_j r_0 - 1}^{(1)})$$

where  $d_j r_k = r_k$  if  $k < j$ ,  $d_j r_k = r_{k+1}$  otherwise, and

$$s_u^k = \begin{cases} t_u^k, & k < j; \\ t_u^j, & k = j \text{ and } u < r_j - r_{j-1} - 1; \\ 1, & k = j \text{ and } u = r_j - r_{j-1} - 1; \\ t_{u-r_j+r_{j-1}}^{j+1}, & k = j \text{ and } u > r_j - r_{j-1} - 1; \\ t_u^{k+1}, & k > j. \end{cases}$$

When  $t_j = 0$ ,  $\min(t_p, t_s) = 0$  whenever  $j \in \{p, s\}$ . So it does not contribute to  $a_u$  and hence we have  $h_n(r)(t(r), t) = h_n(d_j r)(d_j t(r), t')$  where  $d^j t' = t$ . Therefore the maps  $h_n(r)$  induce a map  $h_n: |B(*, \mathcal{F}_\bullet[n], *)| \rightarrow |\Delta^n|$ .

*Proof.* To show that  $h_n$  is onto, take  $(b_0, \dots, b_n) \in \Delta^n$ . Let  $r = (r_0, \dots, r_i)$  where  $r_0$  is the smallest integer for which  $b_u \neq 0$  and  $r_k$ 's are defined inductively as follows. If for all  $u \geq r_k$ ,  $b_{r_k} < b_u$ , define  $r_{k+1} = r_k + 1$ ; otherwise let  $r_{k+1}$  be the largest integer such that  $b_{r_k} \geq b_u$  for all  $r_k < u < r_{k+1}$ . This procedure terminates at  $i$  where  $b_u = 0$  for all  $u > r_i$ . Let  $t_k = \frac{b_{r_k}}{b}$  where  $b = b_{r_0} + \dots + b_{r_i}$  and  $t_j^k = \frac{b_{(r_{k-1}+j)}}{b_{r_{k-1}}}$ . Then  $h_n([t(r), t]) = (b_0, \dots, b_n)$ . Note that we have  $b_{r_0} < b_{r_1} < \dots < b_{r_i}$  by construction and hence  $r = (r_0, r_1, \dots, r_i)$  is the smallest subset of  $\{1, \dots, n\}$  for which  $(b_0, \dots, b_n) \in A_r^n$ .

It remains to show that  $h_n([t(r), t]) = h_n([s(r'), s])$  implies  $(t(r), t) \sim (s(r'), s)$ . For this assume that  $t(r), t$ , and  $r = (r_0, \dots, r_i)$  is defined as above. Since the intersection of  $A_r^n$  and  $A_{r'}^n$  is  $A_{r''}^n$  where  $r'' = r \cap r'$ , we have  $r' \supseteq r$  as a set. So  $r = d_{i_k} \dots d_{i_1} r'$  for some  $i_1 > \dots > i_k$ . Without loss of generality, we can let  $r = d_j r'$ , that is,  $r' = (r_0, \dots, r_{j-1}, r'_j, r_j, \dots, r_i)$ . Once we show that  $s$  is  $(t_0, \dots, t_{j-1}, 0, t_j, \dots, t_i)$ , it immediately follows that  $d_j s(r') = t(r)$ . Therefore we obtain  $(t(r), t) \sim (s(r'), s)$  as desired.

By definition, we have  $a_0 = t_0 = s_0$  and  $a_{r_i} = t_i = s_{i+1}$ . By inductive hypothesis on  $k < j$ ,  $t_l = s_l$  for all  $l \leq m < k$ . Since  $\min(s_m, s_z) < s_m$ , we have

$$s_m = t_m < t_{m+1} = a_{r'_{m+1}} = \max\{s_{m+1}, \max_{m+1 < z} \{\min(s_m, s_z)\}\} = s_{m+1}.$$

Similarly, one can show that  $s_m = t_{m-1}$  for all  $m > j$ . Since the sum of  $s_m$ 's is 1, this forces  $s_j$  to be zero. This proves that the map  $h_n$  is a homeomorphism for each  $n$ .

Since the morphisms in  $\widetilde{\Delta}$  is generated by the face maps  $d^j: [n-1] \rightarrow [n]$  and the degeneracy map  $s^0: [1] \rightarrow [0]$ , to show that  $h$  is natural, it suffices to show that the following diagrams commute

$$\begin{array}{ccccccc} |B(*, \mathcal{F}_\bullet[n-1], *)| & \xrightarrow{d_*^j} & |B(*, \mathcal{F}_\bullet[n], *)| & & |B(*, \mathcal{F}_\bullet[1], *)| & \xrightarrow{s_*^0} & |B(*, \mathcal{F}_\bullet[0], *)| \\ h_{n-1} \downarrow & & h_n \downarrow & & h_1 \downarrow & & h_0 \downarrow \\ \Delta^\bullet([n-1]) & \xrightarrow{d^j} & \Delta^\bullet([n]) & & \Delta^\bullet([1]) & \xrightarrow{s^0} & \Delta^\bullet([0]) \end{array}$$

for every  $0 \leq j \leq n$ . Since  $\Delta^\bullet([0]) \cong |B(*, \mathcal{F}_\bullet[0], *)| \cong *$ , the second diagram commutes. On the other hand, the induced map  $d_*^j: |B(*, \mathcal{F}_\bullet[n-1], *)| \rightarrow |B(*, \mathcal{F}_\bullet[n], *)|$  is explicitly given by

$$d_*^j[t(r), t] = \begin{cases} [t(r), t], & j > r_i; \\ [s(r_+^l), t], & j = r_l, 0 < l \leq i; \\ [p(r_+^{l+1}), t], & r_l < j = u < r_{l+1}, 0 \leq l \leq i; \\ [t(r_+^0), t], & j \leq r_0, \end{cases}$$

where  $r_+^m = (r_0, \dots, r_{m-1}, r_m + 1, \dots, r_i + 1)$  and

$$s_v^m = \begin{cases} 0, & m = l, v = r_l - r_{l-1}, \\ t_v^m, & \text{otherwise,} \end{cases} \quad p_v^m = \begin{cases} 0, & m = l + 1, v = u - r_l, \\ t_{v-1}^m, & m = l + 1, v > u - r_l, \\ t_v^m, & \text{otherwise.} \end{cases}$$

By using this formula, one can easily check that the map  $h_n$  respects the face maps in the sense that it makes the first diagram above commute.  $\square$

**6.2. In general.** Let  $B: \mathbf{Cat} \rightarrow \mathbf{Top}$  be the classifying space functor. Then we have the following result.

**Theorem 6.2.** *There is a natural homeomorphism between the functors  $B\gamma$  and  $|B_\bullet(*, \mathcal{F}_\bullet\gamma, *)|$ .*

*Proof.* First by Section 5.4, the bisimplicial set  $B(*, \mathcal{F}_\bullet\gamma, *)$  is the left Kan extension of  $B(*, \mathcal{F}_\bullet\bar{f}\gamma, *)$  over  $p_\gamma$ . Now by the previous theorem there is a natural homeomorphism between  $|B(*, \mathcal{F}_\bullet\bar{f}\gamma, *)|$  and  $B\bar{f}\gamma$ . We also know  $B\gamma$  is the left Kan extension of  $B\bar{f}\gamma$  over  $p_\gamma$ . Hence we have a natural homeomorphism between the functors  $B\gamma$  and  $|B_\bullet(*, \mathcal{F}_\bullet\gamma, *)|$ .  $\square$

## 7. HOMOTOPY DIAGRAMS

Let  $\mathcal{M}$  be a homotopical category (see [11]), in other words,  $\mathcal{M}$  is a category with a class of morphisms called weak equivalences that contains all the identities and satisfies the *2-out-of-6 property*: if  $hg$  and  $gf$  are weak equivalences, then  $f$ ,  $g$ ,  $h$ , and  $hgf$  are also weak equivalences. Any homotopical category  $\mathcal{M}$  has a homotopy category  $\mathrm{Ho}(\mathcal{M})$ , obtained by formally inverting the weak equivalences. There is a localization functor  $l: \mathcal{M} \rightarrow \mathrm{Ho}(\mathcal{M})$  which is universal among the functors inverting the weak equivalences. Here we have enriched homotopical categories and their homotopy categories have small hom-sets.

**7.1. Homotopy diagrams.** For the following assume that  $\mathcal{G}$  is a small category,  $\gamma: \mathcal{G} \rightarrow \mathbf{Cat}$  is a functor,  $\mathcal{M}$  is a  $\mathbf{cSet}$ -enriched homotopical category with localization functor  $l: \mathcal{M} \rightarrow \mathrm{Ho}(\mathcal{M})$ , and  $f: \int_{\mathcal{G}} \gamma \rightarrow \mathrm{Ho}(\mathcal{M})$  is a functor. Also let  $p_n$  denote the projection from  $\int_{\mathcal{G}} \mathrm{sk}_n^c W\gamma$  to  $\int_{\mathcal{G}} \gamma$ .

**Definition 7.1.** An  $n$ -commutative homotopy  $\gamma$ -diagram over  $f$  is a **cSet**-functor

$$F: \int_{\mathcal{G}} \text{sk}_n^c W\gamma \rightarrow \mathcal{M}$$

which satisfies  $f \circ p_n = l \circ F$ .

**Remark 7.2.** Our definition of a homotopy commutative diagram is in agreement with Vogt's definition [27] when  $\mathcal{M} = \mathbf{Top}_{\mathbf{cSet}}$ . For  $c$  an object of  $\mathcal{G}$  and  $\sigma: [k+1] \rightarrow \gamma(c)$  a functor, let  $f_i$  denote the map  $\sigma(i \rightarrow (i+1))$  for  $0 \leq i \leq k$ ,  $A = F(c, \sigma(0))$ , and  $B = F(c, \sigma(k+1))$ . Then  $F(\text{id}_c, -)$  gives a cubical map from the cube  $I_\sigma$  to  $\mathbf{Top}_{\mathbf{cSet}}(A, B)$ . If we consider this map as a continuous function  $D$  from the  $k$ -cube  $[0, 1]^k$  to  $\mathbf{Top}_{\mathbf{cSet}}(A, B)$  and use the notation  $D(f_k, t_k, \dots, t_0, f_0)$  for the image of  $(t_k, \dots, t_1)$  in  $[0, 1]^k$  then we have the following equations

$$D(f_n, t_n, \dots, f_0) \circ D(g_m, u_m, \dots, g_0) = D(f_n, t_n, \dots, f_0, 1, g_m, u_m, \dots, g_0).$$

and

$$D(f_n, t_n, \dots, f_0) = \begin{cases} D(f_n, t_n, \dots, f_1), & f_0 = \text{id}, \\ D(f_n, t_n, \dots, f_{i+1}, \max\{t_{i+1}, t_i\}, f_{i-1}, \dots, f_0), & f_i = \text{id}, 0 < i < n, \\ D(f_{n-1}, t_{n-1}, \dots, f_0), & f_n = \text{id}, \\ D(f_n, t_n, \dots, t_{i+1}, f_i f_{i-1}, t_{i-1}, \dots, f_0), & t_i = 0. \end{cases}$$

**7.2. Maps between homotopy diagrams.** For a non-negative integer  $k$  let  $\gamma \times [k]$  denote the functor from  $\mathcal{G}$  to **Cat** given by the cartesian product of the functor  $\gamma$  and the constant functor  $[k]$ .

**Definition 7.3.** Let  $n, k$  be two non-negative integers. Then a  $(k-1)$ -map of  $n$ -commutative homotopy  $\gamma$ -diagrams over  $\mathcal{M}$  is a **cSet**-functor

$$F: \int_{\mathcal{G}} \text{sk}_{n+k}^c W(\gamma \times [k]) \rightarrow \mathcal{M}.$$

**Definition 7.4.** Let  $F_1, F_2: \int_{\mathcal{G}} \text{sk}_n^c W\gamma \rightarrow \mathcal{M}$  be two  $n$ -commutative homotopy  $\gamma$ -diagram over  $f$ . Then we say that  $F_1$  is *homotopic* to  $F_2$  if there exists two 1-maps of  $n$ -commutative homotopy  $\gamma$ -diagrams  $H_1, H_2: \int_{\mathcal{G}} \text{sk}_{n+2}^c W(\gamma \times [2]) \rightarrow \mathcal{M}$  such that  $H_i$  restricted to  $\int_{\mathcal{G}} \text{sk}_n^c W(\gamma \times \{j\})$  is the same as  $F_k$  when  $i+j \equiv k \pmod{2}$  and  $H_1$  restricted to  $\int_{\mathcal{G}} \text{sk}_{n+2}^c W(\gamma \times \{j \rightarrow j+1\})$  is the same as  $H_2$  restricted to  $\int_{\mathcal{G}} \text{sk}_{n+2}^c W(\gamma \times \{j' \rightarrow j'+1\})$  when  $j+j'=1$ , and both  $H_1$  and  $H_2$  restricted to  $\int_{\mathcal{G}} \text{sk}_{n+2}^c W(\gamma \times \{0 \rightarrow 2\})$  are induced by the identity.

The following two theorems will explain why such **cSet**-functors are called maps of homotopy diagrams. First we introduce some notations. If  $k$  is a non-negative integer and  $A$  is a subset of  $\{0, 1, 2, \dots, k\}$  then  $\overline{A}$  will denote the full subcategory of  $[k]$  with the object set equal to  $A$  and for a positive integer  $r$ ,  $s_{r, \overline{A}}$  will denote the **cSet**-category

$$s_{r, \overline{A}} = \text{sk}_r^c W(\gamma \times \overline{A})$$

and  $F_{r,\overline{A}}$  will denote the restriction of  $F$  to  $s_{r,\overline{A}}$ . Given  $F$  a  $(k-1)$ -map of  $n$ -commutative homotopy  $\gamma$ -diagram over  $\mathbf{Top}_{\mathbf{cSet}}$ , we will define a map

$$f_k(F): |B_\bullet(*, Ts_{n,\overline{\{0\}}}, F_{n,\overline{\{0\}}})| \times |I^{k-1}| \rightarrow |B_\bullet(*, Ts_{n,\overline{\{k\}}}, F_{n,\overline{\{k\}}})|$$

which satisfies the following two theorems

**Theorem 7.5.** *Let  $F$  be a  $n$ -commutative homotopy  $\gamma$ -diagram over  $\mathbf{Top}_{\mathbf{cSet}}$ . Then  $f_1(F \times [1])$  is homotopy equivalent to the identity map on  $|B_\bullet(*, T \text{sk}_n^c W\gamma, TF)|$  where  $F \times [1]$  denotes the functor obtained by the projection from  $\gamma \times [1]$  to  $\gamma$ .*

The above theorem says up to homotopy identity goes to identity.

**Theorem 7.6.** *The map  $f_k$  also satisfies the following property*

$$f_k(F)(-, i_B) = f_1(F_{n+1, \overline{\{b_{s-1}, b_s\}}}) \circ \cdots \circ f_1(F_{n+1, \overline{\{b_0, b_1\}}})$$

where  $B = \{0 = b_0 < b_1 < \cdots < b_s = k\}$  and  $i_B = (a_1, \dots, a_{k-1})$  in  $|I^{k-1}|$  with  $a_i = 1$  if  $i = b_j$  for some  $1 \leq j \leq s-1$  and  $a_i = 0$  otherwise.

Now to define  $f_k(F)$ , we first give a double bar construction  $B'(*, \tilde{\gamma}, F)$  for  $\tilde{\gamma} = T \text{sk}_{n+k}^c W(\gamma \times [k])$  where  $B'(*, \tilde{\gamma}, F)$  is a functor from  $\mathcal{G}$  to the functor category from  $\Delta \downarrow [k]$  to  $\mathbf{Top}$  defined for  $c$  an object of  $\mathcal{G}$  and  $\sigma: [m] \rightarrow [k]$  in  $\Delta \downarrow [k]$  by

$$B'(*, \tilde{\gamma}, F)(c)(\sigma) = \prod_{\substack{\text{For } 0 \leq i \leq m, \\ \alpha_i \text{ is an object of} \\ \gamma(c) \times \{\sigma(i)\}}} (\tilde{\gamma}(c)(\alpha_{m-1}, \alpha_m) \otimes \cdots \otimes \tilde{\gamma}(c)(\alpha_0, \alpha_1)) \odot F(c, \alpha_0).$$

Let  $i$  be an integer with  $0 \leq i \leq k$ . Then we define functors  $R_i$ ,  $\overline{R}_i$ , and  $\overline{\overline{R}}_i$  from  $\Delta \downarrow [k]$  to  $\mathbf{Top}$  as follows:

$$R_i(\sigma) = \{t_0 e_0 + \cdots t_m e_m \in |\Delta^m| \mid t'_0 = t'_1 = \cdots t'_{i-1} = 0\},$$

$$\overline{R}_i(\sigma) = \{t_0 e_0 + \cdots t_m e_m \in R_i(\sigma) \mid \forall j \geq i, j \neq k \Rightarrow t'_j \leq \frac{1}{2}(1 - \sum_{s=i}^{j-1} t_s)\}, \text{ and}$$

$$\overline{\overline{R}}_{i,t}(\sigma) = \{t_0 e_0 + \cdots t_m e_m \in \overline{R}_i(\sigma) \mid t'_i = t\} \text{ where } t \in [0, \frac{1}{2}]$$

where  $\sigma: [m] \rightarrow [k]$  is a functor and  $t'_j = \sum_{l \in \sigma^{-1}(j)} t_l$  for  $0 \leq j \leq k$ . Notice that for

$0 \leq i \leq k-1$  and  $u$  in  $[0,1]$ , there is a natural transformation  $r_{i,u}$  from  $\overline{R}_i$  to  $\overline{\overline{R}}_i$  which sends  $(t_0 e_0 + \cdots t_m e_m)$  to  $u(t_0 e_0 \cdots t_{d-1} e_{d-1}) + \frac{1-ut'_i}{1-t'_i}(t_d e_d \cdots t_m e_m)$  where  $d$  is the minimum element of the set  $\{\sigma^{-1}(j) \mid i+1 \leq j \leq k\}$ . Note that  $\overline{\overline{R}}_k = R_k$  and  $r_{i,0}(\overline{R}_i) = \overline{R}_{i+1}$ . Hence the composition  $r_{k-1,0} \circ \cdots \circ r_{1,0} \circ r_{0,0}$  induces a map from  $|B'(*, T \text{sk}_{n+k}^c W(\gamma \times [k]), F)|_{\overline{\overline{R}}_0}$  to  $|B'(*, T \text{sk}_n^c W(\gamma \times [k]), F)|_{R_k}$ . To define  $f_k(F)$ , we need the following observations.

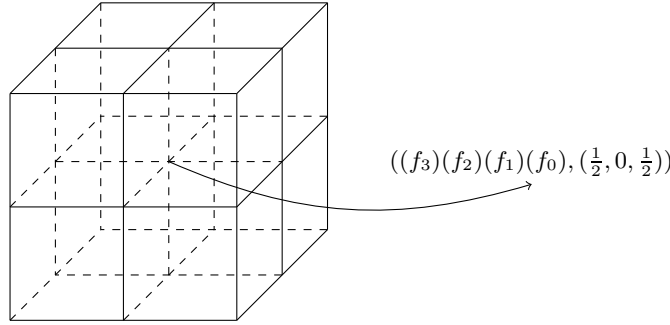
**Lemma 7.7.**  $|B'(*, TW[k], *)|_{\overline{R}_{0, \frac{1}{2}}} \cong I^{k-1}$

*Proof.* Since  $B'(*, TW[k], *)(\sigma) = B'(*, TW[k], *)(s_*^i(\sigma))$ , we can suppress the degenerate objects in  $\Delta \downarrow [k]$ . In addition, if  $\sigma^{-1}(0)$  or  $\sigma^{-1}(k)$  is empty then  $\overline{R}_{0, \frac{1}{2}}(\sigma) = \emptyset$ . Therefore, we have

$$|B'(*, TW[k], *)|_{\overline{R}_{0, \frac{1}{2}}} = \left( \coprod_{\substack{\sigma: [m] \hookrightarrow [k], \\ \sigma(0)=0, \sigma(m)=k}} |C^m(\sigma)| \times \overline{R}_{0, \frac{1}{2}}(\sigma) \right) / \sim$$

where  $|C^m(\sigma)| = |\mathcal{F}_\bullet[k](\sigma(m-1), \sigma(m))| \times \cdots \times |\mathcal{F}_\bullet[k](\sigma(0), \sigma(1))|$ . Note that for every  $\sigma: [m] \hookrightarrow [k]$  with  $\sigma(0) = 0$  and  $\sigma(m) = k$ , we have  $\overline{R}_{0, \frac{1}{2}}(\sigma) \cong I^{m-1}$  and  $|C^m(\sigma)| \cong I^{k-m}$ . Therefore  $|B'(*, TW[k], *)|_{\overline{R}_{0, \frac{1}{2}}}$  is obtained by gluing  $(k-1)$ -cubes

$I_\sigma^{k-1}$ . On the other hand, for each  $m \leq k$ , there are  $\binom{k-1}{m-1}$ -many inclusions  $\sigma: [m] \hookrightarrow [k]$  for which  $\sigma(0) = 0$  and  $\sigma(m) = k$  and for each such  $\sigma$ , the size of the set  $\{(d^i)_*(\sigma) \mid 0 < i < m\}$  is  $(m-1)$ . Moreover there are exactly  $(k-m+1)$ -many  $\gamma: [m+1] \hookrightarrow [k]$  such that  $(d^i)_*(\gamma) = \sigma$  for some  $0 < i < m+1$ . This means that  $I_\sigma^{k-1}$  is glued to  $(k-1)$ -many different cubes. Indeed it is glued to each cube along a different face containing the vertex  $((f_{k-1}) \cdots (f_0), (\frac{1}{2}, 0, \dots, 0, \frac{1}{2}))$ . Therefore, these  $2^{k-1}$ -many  $(k-1)$ -cubes are glued together to form another  $(k-1)$ -cube with center  $((f_{k-1}) \cdots (f_0), (\frac{1}{2}, 0, \dots, 0, \frac{1}{2}))$ .



□

**Lemma 7.8.** *There is a natural inclusion of  $|B_\bullet(*, Ts_{n, \{0\}}, F_{n, \{0\}})| \times |B'(*, TW[k], *)|_{\overline{R}_{0, \frac{1}{2}}}$  in  $|B'(*, T \text{sk}_{n+k}^c W(\gamma \times [k]), F)|_{\overline{R}_0}$ .*

*Proof.* Set  $U = |B'(*, T \text{sk}_{n+k}^c W(\gamma \times [k]), *)|_{\overline{R}_0}$ . We can consider  $U \subseteq B(\gamma \times [k])$  because  $|B'(*, TW(\gamma \times [k]), *)| \cong B(\gamma \times [k])$ . Consider  $V \subseteq B\gamma$ . The projection  $p_1: \gamma \times [k] \rightarrow \gamma$  induces  $Bp_1: B(\gamma \times [k]) \rightarrow B\gamma$ . We have  $Bp_1(U) \subseteq V$ . Define  $f$  as the restriction of  $Bp_1$  to  $U$ . So  $f: U \rightarrow V$ . The projection  $p_2: \gamma \times [k] \rightarrow [k]$  induces  $Wp_2: W(\gamma \times [k]) \rightarrow W[k]$  and this induces  $g: U \rightarrow |B'(*, TW[k], *)|_{\overline{R}_{0, \frac{1}{2}}} \cong I^{k-1}$ . We have  $(f \times g): U \rightarrow V \times I^{k-1}$  an injective map. Define  $X = |B'(*, T \text{sk}_{n+k}^c W(\gamma \times$



$[k]), F)|_{\overline{R}_{0, \frac{1}{2}}}$ . Then  $X$  is a subspace of  $|B'(*, T \operatorname{sk}_{n+k}^c W(\gamma \times [k]), F)|_{\overline{R}_0}$ . Define  $\tilde{F} = F_{n+k, \{0\}}$ . The space  $X$  is same as the pull back of  $|B(*, T \operatorname{sk}_{n+k}^c W\gamma, \tilde{F})| \times I^{k-1} \rightarrow V \times I^{k-1}$  by the map  $f \times g$ . Define  $Z = |B(*, T \operatorname{sk}_n^c W\gamma, *)|$  and consider  $Z \subseteq B\gamma$ . We have  $Z \times I^{k-1} \subseteq (f \times g)(U)$ . Now we are done because the pull back of  $|B(*, T \operatorname{sk}_{n+k}^c W\gamma, \tilde{F})| \times I^{k-1} \rightarrow V \times I^{k-1}$  by the map  $Z \times I^{k-1} \rightarrow V \times I^{k-1}$  is  $|B_\bullet(*, Ts_{n, \overline{\{0\}}}, F_{n, \overline{\{0\}}})| \times |B'(*, TW[k], *)|_{\overline{R}_{0, \frac{1}{2}}}$ .  $\square$

**Lemma 7.9.** *There is a natural homeomorphism between  $|B'(*, \operatorname{sk}_n^c W(\gamma \times [k]), F)|_{R_k}$  and  $|B_\bullet(*, Ts_{n, \overline{\{k\}}}, F_{n, \overline{\{k\}}})|$ .*

*Proof.* For each  $[n] \in \Delta$ , let  $\sigma_{[n]}: [n] \rightarrow [k]$  be defined by  $\operatorname{Im}(\sigma_{[n]}) = \{k\}$ . Then  $p: \Delta \rightarrow \Delta \downarrow [k]$  defined by  $p([n]) = \sigma_n$  is a fibration with  $R_k(p([n])) = |\Delta^n|$ . Since we have  $B'(*, \operatorname{sk}_n^c W(\gamma \times [k]), F)(c)(\sigma_{[n]}) = B_\bullet(*, Ts_{n, \overline{\{k\}}}, F_{n, \overline{\{k\}}})(c)([n])$ , the result follows from Proposition 4.2.  $\square$

Now we can define the map  $f_k(F)$  as the composition of the homeomorphism  $|B_\bullet(*, Ts_{n, \overline{\{0\}}}, F_{n, \overline{\{0\}}})| \times |I^{k-1}| \cong |B_\bullet(*, Ts_{n, \overline{\{0\}}}, F_{n, \overline{\{0\}}})| \times |B'(*, TW[k], *)|_{\overline{R}_{0, \frac{1}{2}}}$  and the inclusion of  $|B_\bullet(*, Ts_{n, \overline{\{0\}}}, F_{n, \overline{\{0\}}})(c)| \times |B'(*, TW[k], *)|_{\overline{R}_{0, \frac{1}{2}}}$  in the realization  $|B'(*, T \operatorname{sk}_{n+k}^c W(\gamma \times [k]), F)|_{\overline{R}_0}$  and the above map to  $|B'(*, \operatorname{sk}_n^c W(\gamma \times [k]), F)|_{R_k}$  which is homeomorphic to the space  $|B_\bullet(*, Ts_{n, \overline{\{k\}}}, F_{n, \overline{\{k\}}})|$ .

Note that Theorem 7.5 follows directly from the construction of the map  $f_1$  and the fact that  $r_{0,0}$  is a deformation retract. On the other hand, the cubes in the proof of Lemma 7.7 are glued together to form the  $(k-1)$ -cube  $I^{k-1}$  as follow. For each  $\sigma: [m] \hookrightarrow [k]$  with  $\sigma(0) = 0$  and  $\sigma(m) = k$ , there is one cube  $I_\sigma^{k-1}$  and each such cube is glued to others except the three faces passing through a vertex which forms a vertex  $v = \{v_1, \dots, v_{k-1}\}$  of  $I^{k-1}$  with  $v_j = 1$  if  $j$  is in the image of  $\sigma$ . Therefore, Theorem 7.6 directly follows.

## 8. OBSTRUCTIONS

**8.1. Bredon cohomology.** In this section, assume that  $\mathcal{G}$  is a groupoid and  $\mathbf{Ab}$  denotes the category of abelian groups. Let  $\gamma: \mathcal{G} \rightarrow \mathbf{Cat}$  be a functor. We write  $d\gamma$  for the composition of  $\gamma$  with  $(\Delta \downarrow \cdot): \mathbf{Cat} \rightarrow \mathbf{Cat}$ . We define

$$I\gamma = \int_{\mathcal{G}} d\gamma.$$

**Definition 8.1.** A local coefficient system on  $\gamma$  is a functor  $M: I\gamma \rightarrow \mathbf{Ab}$ .

Let  $D$  denote the projection from  $I\gamma$  to  $\Delta$ . Given a local coefficient system  $M$  on  $\gamma$  we obtain a cosimplicial abelian group  $\overline{M}: \Delta \rightarrow \mathbf{Ab}$  as the right Kan extension of  $M$  over  $\pi$ . Let  $C_{\mathcal{G}}^*(\gamma; M)$  denote the associated cochain complex to  $\overline{M}$ . In other

words

$$C_{\mathcal{G}}^*(\gamma; \mathcal{M}) = \lim_{\mathcal{G}} C^*(\gamma; \mathcal{M})$$

where  $C^*(\gamma; \mathcal{M})$  is considered as a functor from  $\mathcal{G}$  to the category of cochain complexes defined as follows:

$$C^n(\gamma; \mathcal{M})(c) = \prod_{\sigma: [n] \rightarrow \gamma(c)} \overline{M}(c, \sigma)$$

for  $c$  an object of  $\mathcal{G}$  and

$$\delta_c(f)(\tau) = \sum_i (-1)^i \overline{M}(\text{id}_c, d^i)(f(\tau \circ d^i))$$

for  $f$  in  $C^n(\gamma; \mathcal{M})(c)$  and  $\tau: [n+1] \rightarrow \gamma(c)$ . For  $g: c \rightarrow c'$  a morphism in  $\mathcal{G}$ , a map between  $C^n(\gamma; \mathcal{M})(c)$  and  $C^n(\gamma; \overline{M})(c')$  is defined by  $(gf)(\sigma) = \overline{M}((g, \text{id}_\sigma))(f(g^{-1}))$ .

**Definition 8.2.** The Bredon cohomology of  $\gamma$  with coefficients in  $\mathcal{M}$  is defined as the cohomology of the cochain complex  $C_{\mathcal{G}}^*(\gamma; \mathcal{M})$ .

To define relative Bredon cohomology take  $\alpha = (\alpha_i)_{i \in \mathcal{I}}$  where  $\alpha_i: \mathcal{G} \rightarrow \mathbf{Cat}$  is a functor such that the category  $\alpha(c)$  is a subcategory of  $\gamma(c)$  for every object  $c$  of  $\mathcal{G}$  and the inclusions give a natural transformation from  $\alpha_i$  to  $\gamma$  for each  $i$  in  $\mathcal{I}$ . Let  $\mathcal{M}_{\alpha_i}$  denote the restriction of  $\mathcal{M}$  to  $\int_{\mathcal{G}} \alpha_i$  for  $i$  in  $\mathcal{I}$ . Notice that  $C_{\mathcal{G}}^*(\alpha_i; \mathcal{M}_{\alpha_i})$  can be considered as a subcochain complex of  $C_{\mathcal{G}}^*(\gamma; \mathcal{M})$ . Let  $C_{\mathcal{G}}^*(\alpha; \mathcal{M}_{\alpha})$  denote cochain complex which is generated by the cochain complexes  $C_{\mathcal{G}}^*(\alpha_i; \mathcal{M}_{\alpha_i})$  as  $i$  ranges over  $\mathcal{I}$ . Hence we obtain a relative cochain complex

$$C_{\mathcal{G}}^n(\gamma, \alpha; \mathcal{M}) = C_{\mathcal{G}}^n(\gamma; \mathcal{M}) / C_{\mathcal{G}}^n(\alpha; \mathcal{M}_{\alpha})$$

for  $n$  an integer.

**Definition 8.3.** The relative Bredon cohomology of the pair  $(\gamma, \alpha)$  with coefficients in  $\mathcal{M}$  is defined as the cohomology of the cochain complex  $C_{\mathcal{G}}^*(\gamma, \alpha; \mathcal{M})$ .

Let  $d_n\gamma = \pi^{-1}([n])$  be the subcategory of  $\int_{\mathcal{G}} d\gamma$  with objects which are sent to  $[n]$  by  $\pi$  and morphisms which are sent to identity by  $\pi$ . We can write  $d_n\gamma$  as a disjoint union of two full subcategories  $Nd_n\gamma$  and  $Dd_n\gamma$  where an object  $(c, \sigma)$  of  $d_n\gamma$  is in  $Nd_n\gamma$  if  $\sigma$  is not in the image of a degeneracy map in the category  $d\gamma(c)$  and the image of  $\sigma$  is not in  $\alpha_i(c)$  for any  $i$  in  $\mathcal{I}$  and an object  $(c, \sigma)$  of  $d_n\gamma$  is in  $Dd_n\gamma$  otherwise. Now we can define a subcochain complex of  $C_{\mathcal{G}}^*(\gamma; \mathcal{M})$  as follows:

$$DC_{\mathcal{G}}^n(\gamma, \alpha; \mathcal{M}) = \lim_{(c, \sigma) \in Dd_n\gamma} M(c, \sigma).$$

This cochain complex contains  $C_{\mathcal{G}}^*(\alpha; \mathcal{M}_{\alpha})$ . Hence we can define a new cochain complex

$$\overline{C}_{\mathcal{G}}^n(\gamma, \alpha; \mathcal{M}) = C_{\mathcal{G}}^n(\gamma; \mathcal{M}) / DC_{\mathcal{G}}^n(\gamma, \alpha; \mathcal{M}_{\alpha}).$$

The relative Bredon cohomology of the pair  $(\gamma, \alpha)$  with coefficients in  $\mathcal{M}$  is isomorphic to the cohomology of the cochain complex  $\overline{C}_{\mathcal{G}}^*(\gamma, \alpha; \mathcal{M})$ . We can also define a subgroup of  $C_{\mathcal{G}}^n(\gamma; \mathcal{M})$  as follows:

$$NC_{\mathcal{G}}^n(\gamma, \alpha; \mathcal{M}) = \lim_{(c, \sigma) \in Nd_n \gamma} M(c, \sigma).$$

Now notice that as an abelian group  $\overline{C}_{\mathcal{G}}^n(\gamma, \alpha; \mathcal{M})$  is isomorphic to  $NC_{\mathcal{G}}^n(\gamma, \alpha; \mathcal{M})$  because we have

$$C_{\mathcal{G}}^*(\gamma, \alpha; \mathcal{M}) = NC_{\mathcal{G}}^n(\gamma, \alpha; \mathcal{M}) \oplus DC_{\mathcal{G}}^n(\gamma, \alpha; \mathcal{M}).$$

**8.2. Obstructions to extending homotopy diagrams.** Assume that we have a **cSet**-functor

$$F: \int_{\mathcal{G}} \text{sk}_k^c(W\gamma, \text{sk}_{k+r}^c W\alpha) \rightarrow \mathcal{M}$$

where  $k$  and  $r$  are positive integers. For  $c$  an object of  $\mathcal{G}$  and  $\sigma: [n] \rightarrow \gamma(c)$  an object of  $d\gamma(c)$ , let  $M_F(c, \sigma)$  denote the path component of  $\text{mor}_{\mathcal{M}}(F(c, \sigma(0)), F(c, \sigma(n)))$  which contains the morphism  $F(\text{id}_c, \sigma(0 \rightarrow n))$  and  $\mathcal{A}_{\gamma}(c, \sigma)$  denotes the automorphism group of the object  $(c, \sigma)$  in the category  $\pi^{-1}([n])$ . Then we define a functor

$$\mathbf{m}F: \int_{\mathcal{G}} d\gamma \rightarrow \mathbf{cSet}$$

such that for  $(c, \sigma)$  an object of  $\int_{\mathcal{G}} d\gamma$

$$\mathbf{m}F(c, \sigma) = \lim_{\mathcal{A}_{\gamma}(c, \sigma)} M_F(c, \sigma)$$

and for a morphism  $(g, 1)$  in  $\int_{\mathcal{G}} d\gamma$ , the morphism  $\mathbf{m}F(g, 1): \mathbf{m}F(c, \sigma) \rightarrow \mathbf{m}F(c', \gamma(g)\sigma)$  is induced by the morphisms

$$F(g^{-1}, 1): F(c', \gamma(g)\sigma(0)) \rightarrow F(c, \sigma(0)) \text{ and } F(g, 1): F(c, \sigma(n)) \rightarrow F(c, \gamma(g)\sigma(n))$$

and for  $\sigma: [n] \rightarrow \gamma(c)$  and  $\tau: [m] \rightarrow \gamma(c)$  two objects of  $d\gamma(c)$  and  $f: [n] \rightarrow [m]$  a morphism from  $\sigma$  to  $\tau$  in  $d\gamma(c)$ , the morphism  $\mathbf{m}F(1, f): \mathbf{m}F(c, \sigma) \rightarrow \mathbf{m}F(c, \tau)$  is induced by the morphisms

$$F(1, \tau(0 \rightarrow 1)) \circ \cdots \circ F(1, \tau((f(0) - 1) \rightarrow f(0))): F(c, \tau(0)) \rightarrow F(c, \sigma(0))$$

and

$$F(1, \tau(f(n) \rightarrow (f(n) + 1))) \circ \cdots \circ F(1, \tau((m - 1) \rightarrow m)): F(c, \sigma(n)) \rightarrow F(c, \tau(m)).$$

Now we will discuss obstructions to extending a relative  $k$ -commutative homotopy diagram  $F$ , in other words, we want to define obstructions to the existence of a **cSet**-functor from  $\int_{\mathcal{G}} \text{sk}_{k+1}^c(W\gamma, \text{sk}_{k+r}^c W\alpha)$  to  $\mathcal{M}$  which is equal to  $F$  when restricted to  $\int_{\mathcal{G}} \text{sk}_k^c(W\gamma, \text{sk}_{k+r}^c W\alpha)$ . Let  $I^{k+1}$  and  $\partial I^{k+1}$  denote the constant functors from  $\int_{\mathcal{G}} d_{k+2}\gamma$  to **cSet** constantly equal to the standard  $(k+1)$ -cube and its boundary

respectively. We define a natural transformation  $o_F^{k+1}$  from  $\partial I^{k+1}$  to  $\mathbf{m}F$  restricted to  $d_{k+2}\gamma$  as follows:

$$o_F^{k+1}(c, \sigma): \partial I^{k+1} \cong \partial (\text{mor}_{W[k+2]}(0, k+2)) \xrightarrow{F(\text{id}_c, \cdot) \circ W\sigma} mF(c, \sigma)$$

for  $c$  an object of  $\mathcal{G}$  and  $\sigma: [k+2] \rightarrow \gamma(c)$  an object of  $d_{k+2}\gamma(c)$ . Assuming  $\mathbf{m}F(c, \sigma)$  is fibrant and simple for every object  $(c, \sigma)$  of  $\int_{\mathcal{G}} d\gamma$ , we can define a local coefficient system on  $\gamma$

$$\pi_k F: \int_{\mathcal{G}} d\gamma \rightarrow \mathbf{Ab}$$

by composing  $\mathbf{m}F$  with  $\pi_k$ . Now we can consider  $o_F^{k+1}$  as an element in  $\overline{\mathcal{C}}_{\mathcal{G}}^{k+2}(\gamma, \alpha; \pi_k F)$ . Now the following result shows that  $o_F^{k+1}$  can be considered as an obstruction to extension.

**Proposition 8.4.** *A  $\mathbf{cSet}$ -functor  $F: \int_{\mathcal{G}} \text{sk}_k^c(W\gamma, \text{sk}_{k+r}^c W\alpha) \rightarrow \mathcal{M}$  lifts to a  $\mathbf{cSet}$ -functor  $F': \int_{\mathcal{G}} \text{sk}_{k+1}^c(W\gamma, \text{sk}_{k+r}^c W\alpha) \rightarrow \mathcal{M}$  if and only if  $o_F^{k+1} = 0$ .*

*Proof.* We have

$$o_F^{k+1}(c, \sigma) = 0 \in \pi_k F(c, \sigma)$$

for all  $(c, \sigma)$  in  $Dd_{k+2}\gamma$ . Moreover we have

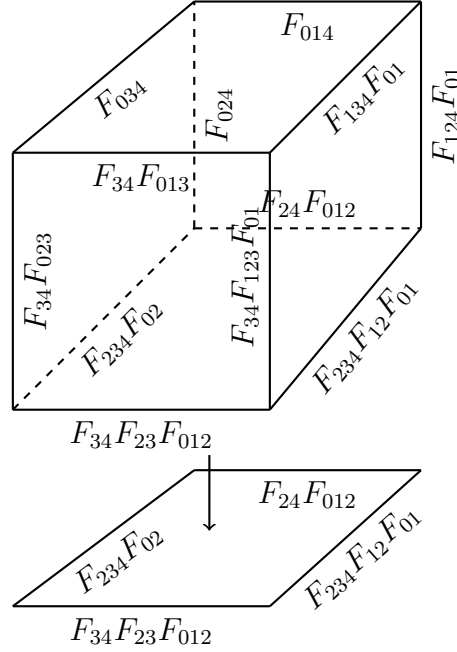
$$\pi_k \left( \lim_{(c, \sigma) \in Nd_{k+2}\gamma} mF(c, \sigma) \right) \cong \lim_{(c, \sigma) \in Nd_{k+2}\gamma} \pi_k F(c, \sigma).$$

Hence the result follows. □

Now we show that  $o_F^{k+1}$  is a cocycle.

**Lemma 8.5.**  $\delta o_F^{k+1} = 0$ .

For  $k = 1$  and  $\tau: [4] \rightarrow \gamma(c)$  an object of  $d_4\gamma(c)$ ,  $(\delta o_F^2)(c, \tau)$  is represented by the following cube:



where  $F_{i_1 i_2 \dots i_s}$  denotes the map  $F(\text{id}_c, \tau(i \rightarrow i_2 \dots \rightarrow i_s))$ . As shown in the picture,  $(\delta o_F^2)(\text{id}_c, \tau)$  is also represented by the boundary of the below square. Since the square can be filled by the map

$$F_{234}F_{012}: I^2 \rightarrow mF(c, \tau),$$

it vanishes.

*Proof.* First notice that for any  $c$  an object of  $\mathcal{G}$  and  $\tau: [k+3] \rightarrow \gamma(c)$  an object of  $d_{k+3}\gamma(c)$ , we have

$$\sum_{i=1}^{k+2} \sum_{\epsilon=0}^1 (-1)^{i+\epsilon} F(\text{id}_c, \cdot) \circ W\tau \circ (d^{(i,\epsilon)}|_{\partial I^{k+1}}) = 0$$

in the abelian group  $[\partial I^{k+1}, \mathbf{m}F(c, \tau)]$  where we consider the face map  $d^{(i,\epsilon)}$  from  $I^{k+1}$  to  $I^{k+2} \cong \text{mor}_{W[k+3]}(0, k+3)$  and let  $d^{(i,\epsilon)}|_{\partial I^{k+1}}$  denote the restriction of the face map  $d^{(i,\epsilon)}$  to  $\partial I^{k+1}$ . Assume that  $A(i, \epsilon)$  denote  $F(\text{id}_c, \cdot) \circ W\tau \circ (d^{(i,\epsilon)}|_{\partial I^{k+1}})$ . Then

$$\delta o_F^{k+1}(c, \tau) = \sum_{i=1}^{k+2} (-1)^i A(i, 0) - A(0, 1) + A(k+2, 1) = \sum_{i=2}^{k+1} (-1)^i A(i, 1) = 0.$$

□

To show that  $[o_F^{k+1}]$  is a cohomological obstruction for lifting the **cSet**-functor  $F$  to a **cSet**-functor  $F'$ :  $\int_{\mathcal{G}} \text{sk}_{k+1}^c(W\gamma, \text{sk}_{k+r}^c W\alpha) \rightarrow \mathcal{M}$ , we first note that there is an action of  $\pi_k(\mathbf{m}F(c, \sigma))$  on homotopy classes of maps from  $I^k$  to  $\mathbf{m}F(c, \sigma)$  relative to  $\partial I^k$  since  $\mathbf{m}F(c, \sigma)$  is fibrant and simple for every object  $(c, \sigma)$  of  $\int_{\mathcal{G}} d\gamma$ . Let  $f \cdot g$

denote the action of  $g$  in  $\pi_k(\mathbf{m}F(c, \sigma))$  on  $f$  a homotopy class of a map from  $I^k$  to  $\mathbf{m}F(c, \sigma)$  relative to  $\partial I^k$ .

**Proposition 8.6.** *Let  $F: \int_{\mathcal{G}} \text{sk}_k^c(W\gamma, \text{sk}_{k+r}^c W\alpha) \rightarrow \mathcal{M}$  be a  $\mathbf{cSet}$ -functor. Then the restriction of  $F$  to a  $(k-1)$ -th skeleton lifts to a  $\mathbf{cSet}$ -functor on  $\int_{\mathcal{G}} \text{sk}_{k+1}^c(W\gamma, \text{sk}_{k+r}^c W\alpha)$  if and only if  $[o_F^{k+1}] = 0$  in  $H^{k+2}(\gamma, \alpha; \pi_k F)$ .*

*Proof.* We have  $[o_F^{k+1}] = 0$  in  $H^{k+2}(\gamma, \alpha; \pi_k F)$ . Hence there exists  $g$  in  $\overline{C}_{\mathcal{G}}^{k+1}(\gamma, \alpha; \pi_k F)$  such that  $\delta g = o_F^{k+1}$ . The restriction of  $F$  to  $(k-1)$ -skeleton can be extended to a  $\mathbf{cSet}$ -functor on  $\int_{\mathcal{G}} \text{sk}_k^c(W\gamma, \text{sk}_{k+r}^c W\alpha)$  denoted by  $\tilde{F}$  so that  $\tilde{F}|_{I_{\sigma}^n}$  is equal to  $(F|_{I_{\sigma}^n}) \cdot g(c, \sigma)^{-1}$  as a map from  $I_{\sigma}^n$  to  $\mathbf{m}F(c, \sigma)$  for  $(c, \sigma)$  in  $Nd_{k+2}\gamma(c)$ . Hence  $o_{\tilde{F}}^{k+1} = 0$  and  $\tilde{F}$  lifts to a  $\mathbf{cSet}$ -functor on  $\int_{\mathcal{G}} \text{sk}_{k+1}^c(W\gamma, \text{sk}_{k+r}^c W\alpha)$ .  $\square$

## 9. APPLICATIONS

**9.1. Equivariant homotopy.** Let  $\mathcal{M}$  be a category and  $\mathcal{G}$  be a groupoid. A  $\mathcal{G}$ -object of  $\mathcal{M}$  is a functor from  $\mathcal{G}$  to  $\mathcal{M}$ . Let  $G$  be a group and  $\mathcal{G}_G$  be the associated groupoid. Then a  $G$ -object of  $\mathbf{Top}$  can be also called a  $G$ -space, a  $G$ -object of  $\mathbf{sSet}$  can be considered as a simplicial set with a  $G$  action, and a  $G$ -object of  $\mathbf{CW}$  can be called a  $G$ -CW-complex. Let  $\mathcal{F}$  be a family of subgroupoids of  $\mathcal{G}$ . We say  $\mathcal{G}$ -object  $X: \mathcal{G} \rightarrow \mathcal{M}$  has isotropy in  $\mathcal{F}$  if for all  $\mathcal{H}$  subgroupoid of  $\mathcal{G}$  not in  $\mathcal{F}$  the limit  $\lim_{\mathcal{H}} X$  is an initial object.

Given  $A, B$  two  $\mathcal{G}$ -objects of  $\mathcal{M}$ , a  $\mathcal{G}$ -map from  $A$  to  $B$  is a natural transformation from  $A$  to  $B$ . Further assume that  $\mathcal{M}$  is a model category. Let  $f, g$  be two  $\mathcal{G}$ -maps from  $A$  to  $B$  we say  $f$  is homotopic to  $g$  if there exists a cylinder object  $I$  with two trivial cofibrations  $i_0: * \rightarrow I$  and  $i_1: * \rightarrow I$  and a natural transformation  $H$  from  $A \times I$  to  $B$  such that  $f = H \circ i_0$  and  $g = H \circ i_1$ . So one can also define homotopy equivalence between  $\mathcal{G}$ -objects. We call such  $\mathcal{G}$ -maps  $\mathcal{G}$ -homotopy equivalences.

**9.2. Equivariant classifying spaces.** Let  $\mathcal{G}$  be a groupoid and  $X: \mathcal{G} \rightarrow \mathbf{sSet}$  be a functor. We define  $Sd\gamma: \mathcal{G} \rightarrow \mathbf{Cat}$  to be the functor which send the object  $c$  to  $Sd\gamma(c)$ , the opposite of the full subcategory of  $\Delta \downarrow Z$  with object set equal to non-degenerate objects. Assume  $N$  denotes the nerve functor from  $\mathbf{Cat}$  to  $\mathbf{sSet}$ . Then there is a natural homeomorphism between the realizations  $|X|$  and  $|NSd(X)|$ .

Let  $\mathcal{F}$  be a family of subgroups of a group  $G$  which is closed under conjugation. Define  $\mathcal{E}_{\mathcal{F}}$  as the functor from  $\mathcal{G}_G$  to the category  $\mathbf{Cat}$  which sends the object  $*_G$  in  $\mathcal{G}_G$  to the category  $\mathcal{E}_{\mathcal{F}}(*_G)$  whose objects are pairs  $(G/H, xH)$  where  $H \in \mathcal{F}$  and  $x \in G$  and morphisms from  $(G/H, xH)$  to  $(G/K, yK)$  are the  $G$ -maps from  $G/H$  to  $G/K$  which sends  $xH$  to  $yK$ . Moreover  $g \in G$  sends the object  $(G/H, xH)$  to the object  $(G/H, gxH)$  and sends the morphism  $f: G/H \rightarrow G/K$  to the morphism  $gf: G/H \rightarrow G/K$  which satisfies  $(gf)(gxH) = gyK$ . The functor  $\mathcal{E}_{\mathcal{F}}$  has the following universal property

**Proposition 9.1.** *For any functor  $X: \mathcal{G}_G \rightarrow \mathbf{sSet}$  with isotropy in  $\mathcal{F}$  there exists a natural transformation  $f_X$  from  $SdX$  to  $\mathcal{E}_{\mathcal{F}}$ .*

*Proof.* Let  $X$  be a functor from  $\mathcal{G}_G$  to  $\mathbf{sSet}$ . We want to define a natural transformation from  $Sd(X)$  to  $\mathcal{E}_{\mathcal{F}}$ . This means that we want to define a functor from  $Sd(X)(*_G)$  to  $\mathcal{E}_{\mathcal{F}}(*_G)$ . Choose  $I$  a set of representatives from each orbit of the action of  $G$  on the set of objects in  $Sd(X)(*_G)$ . Send  $x$  in  $I$  to  $(G/G_x, G_x)$  in  $\mathcal{E}_{\mathcal{F}}(*_G)$  and extend the functor equivariantly on the rest of the objects in  $Sd(X)(*_G)$ . Then there is a unique way to send the morphisms.  $\square$

**9.3. Pullbacks.** Let  $G$  be a group and  $\mathcal{F}$  be a family of subgroups of  $G$ , and let  $X: \mathcal{G}_G \rightarrow \mathbf{sSet}$  be a  $G$ -object of  $\mathbf{sSet}$  with isotropy in  $\mathcal{F}$ . By the above section we know that there exists a natural transformation  $f$  from  $SdX$  to  $\mathcal{E}_{\mathcal{F}}$ . This induces a natural transformation from  $T \operatorname{sk}_n^c W SdX$  to  $T \operatorname{sk}_n^c W \mathcal{E}_{\mathcal{F}}$ . Therefore given

$$F: \int_{\mathcal{G}_G} \operatorname{sk}_n^c W \mathcal{E}_{\mathcal{F}} \rightarrow \mathcal{M}$$

we can define the pullback of  $F$  over  $X$  as follows:

$$P_f(X, F) = B(*, T \operatorname{sk}_n^c W SdX, f^*(TF)).$$

Since we could also pull back homotopies we obtain the following result.

**Theorem 9.2.** *If  $F_1$  and  $F_2$  are homotopic as  $n$ -commutative diagrams over  $\mathbf{Top}_{\mathbf{cSet}}$  then  $|P_f(X, F_1)|$  and  $|P_f(X, F_2)|$  are homotopic as  $G$ -objects of  $\mathbf{Top}$ .*

*Proof.* By definition  $F_1$  and  $F_2$  are homotopic as  $n$ -commutative diagrams over  $\mathbf{Top}_{\mathbf{cSet}}$  means there exists  $H_1, H_2: \int_{\mathcal{G}} \operatorname{sk}_{n+2}^c W(\gamma \times [2]) \rightarrow \mathcal{M}$  which satisfies certain conditions which guarantees that  $H_1$  restricted to  $\int_{\mathcal{G}} \operatorname{sk}_{n+1}^c W(\gamma \times \{0 \rightarrow 1\})$  induces a map from  $|P_f(X, F_1)|$  to  $|P_f(X, F_2)|$  and  $H_1$  restricted to  $\int_{\mathcal{G}} \operatorname{sk}_{n+1}^c W(\gamma \times \{1 \rightarrow 2\})$  induces a map from  $|P_f(X, F_2)|$  to  $|P_f(X, F_1)|$  and the compositions of these maps are  $G$ -homotopic to identity by Theorem 7.5 and Theorem 7.6.  $\square$

**9.4. Finiteness.** Assume

$$F: \int_{\mathcal{G}_G} \operatorname{sk}_n^c W \mathcal{E}_{\mathcal{F}} \rightarrow \mathbf{Top}_{\mathbf{cSet}}$$

with isotropy in  $\mathcal{F}$ . Let  $U: \mathbf{CW} \rightarrow \mathbf{Top}$  denote the functor that sends a CW-complex to its underlying topological space. Here  $\mathbf{CW}$  can be considered as a homotopical category with weak equivalences as cellular maps which induce isomorphisms on homotopy groups and  $U$  can be considered as a homotopical functor. We will say a simplicial set is *finite* if it has finitely many non-degenerate simplices.

**Theorem 9.3.** *If  $X$  is finite simplicial set and  $F$  restricted to  $\int_{\mathcal{G}_G} \operatorname{sk}_{-1}^c W \mathcal{E}_{\mathcal{F}}$  factors through  $\mathbf{CW}$  then  $|P_f(X, F)|$  has isotropy in  $\mathcal{F}$  and it is  $G$ -homotopy equivalent to a finite  $G$ -CW-complex.*

*Proof.* Let  $g$  be a map from  $A$  to  $|X|$ . Define  $B_g$  as the pull back of the diagram  $A \rightarrow |X| \leftarrow |P_f(X, F)|$  where the map on the left is  $g$  and the map on the right is the composition  $|P_f(X, F)| \rightarrow |P_f(X, *)| \hookrightarrow |X|$ . For  $i \geq 0$ , define  $A_i = |\text{sk}_i X|$ . We have an inclusion  $g_i : A_i \rightarrow |X|$ . Since  $F$  restricted to  $\int_{\mathcal{G}_G} \text{sk}_{-1}^c W\mathcal{E}_{\mathcal{F}}$  factors through **CW** we can say that  $B_{g_0} = |P_f(\text{sk}_0 X, F)|$  is  $G$ -homotopy equivalent to a finite  $G$ -CW-complex. Assume that we proved  $B_{g_i}$  is  $G$ -homotopy equivalent to a finite  $G$ -CW-complex. Define  $\bar{A}_{i+1}$  as  $|\text{sk}_{i+1} X|$  with a smaller open disc removed from every open  $(i+1)$ -cell. There is an inclusion  $\bar{g}_{i+1} : \bar{A}_{i+1} \rightarrow |X|$ . Define  $\bar{\bar{A}}_{i+1}$  as disjoint union of the closures of the smaller open discs removed above. There is an inclusion  $\bar{\bar{g}}_{i+1} : \bar{\bar{A}}_{i+1} \rightarrow |X|$ . Now  $B_{g_{i+1}}$  is equal to the union of  $B_{\bar{g}_{i+1}}$  and  $B_{\bar{\bar{g}}_{i+1}}$ . The space  $B_{\bar{g}_{i+1}}$  is homotopy equivalent to  $B_{g_i}$  hence, it is  $G$ -homotopy equivalent to a finite  $G$ -CW-complex. The space is a disjoint union of products of finite  $G$ -CW-complexes. Moreover the inclusion of the intersection of  $B_{\bar{g}_{i+1}}$  and  $B_{\bar{\bar{g}}_{i+1}}$  in  $B_{\bar{\bar{g}}_{i+1}}$  is a cofibration. Hence  $B_{g_{i+1}}$  is  $G$ -homotopy equivalent to a finite  $G$ -CW-complex. Hence by induction  $|P_f(X, F)|$  is  $G$ -homotopy equivalent to a finite  $G$ -CW-complex. Moreover  $|P_f(X, F)|$  has isotropy in  $\mathcal{F}$  as it is obtained as a pullback of a space which has isotropy in  $\mathcal{F}$ .  $\square$

**9.5. Group actions on products of spheres.** Let  $G$  be a finite group and let  $X$  be a  $G$ -object of **sSet** with all isotropies in  $\mathcal{F}$  where  $\mathcal{F}$  is a family of subgroups of  $G$  which is closed under conjugation and taking subgroups. Suppose also that  $X$  is a finite  $G$ -CW-complex with  $\dim X = k$  (that is the dimension of  $X(*_G)$ ). Then the standard geometric realization  $|X|$  of  $X$  is a finite  $G$ -CW-complex.

A family of representations  $\{\alpha_H : H \rightarrow U(m)\}_{H \in \mathcal{F}}$  is called *compatible* if for every map  $c_g : H \rightarrow K$  induced by conjugation  $c_g(h) = ghg^{-1}$ , there is a  $\gamma \in U(m)$  such that the following diagram commutes

$$\begin{array}{ccc} H & \xrightarrow{\alpha_H} & U(m) \\ c_g \downarrow & & \downarrow c_\gamma \\ K & \xrightarrow{\alpha_K} & U(m). \end{array}$$

Given a family of representations  $\{\alpha_H\}_{H \in \mathcal{F}}$ , we will construct a  $k$ -commutative homotopy  $\mathcal{E}_{\mathcal{F}}$ -diagram  $F_1 : \int_{\mathcal{G}_G} \text{sk}_k^c W\mathcal{E}_{\mathcal{F}} \rightarrow \mathbf{Top}_{\mathbf{cSet}}$  such that  $F_1|_{\text{sk}_k^c W\mathcal{E}_{\mathcal{F}}(*_G)}$  is homotopic to the  $k$ -commutative homotopy diagram  $F_2 : \text{sk}_k^c W\mathcal{E}_{\mathcal{F}} \rightarrow \mathbf{Top}_{\mathbf{cSet}}$  which sends every object to  $\mathbb{S}^q$  and every morphism to the identity map on  $\mathbb{S}^q$  for some  $q \gg 0$ . Since  $|P_f(X, F_2)| = |X| \times \mathbb{S}^q$ , the following result will follow by Theorem 9.2 and Theorem 9.3.

**Theorem 9.4.** *Let  $\{\alpha_H\}_{H \in \mathcal{F}}$  be a compatible family of representations where  $\mathcal{F}$  is a family of subgroups of  $G$  which is closed under conjugation and taking subgroups.*



Then for every finite  $G$ -object  $X$  of  $\mathbf{sSet}$  with all isotropies in  $\mathcal{F}$ , there is a finite  $G$ -CW-complex  $Y \cong |X| \times \mathbb{S}^q$  whose isotropy subgroups are in the family  $\mathcal{V} = \{H_x | x \in \mathbb{S}(\alpha_H), H \in \mathcal{F}\}$  for some  $q \gg 0$ .

Let  $X$ ,  $\mathcal{F}$  and  $\{\alpha_H\}_{H \in \mathcal{F}}$  be defined as in the above theorem with  $\dim(X) = k$ . For every  $H \in \mathcal{F}$  and  $u \in G/H$ , fix  $x_{(H,u)} \in u$ . We write  $x = x_{H,u}$  when it is clear from the context. Let  $X_H = (G \times \mathbb{S}^{2m-1}) / \sim$  where  $(gh, s) = (g, \alpha_H(h)s)$  and  $X_{H,u} = \{[x_{H,u}, s] \in X_H\}$ . The group  $G$  acts on  $X_H$  by  $g'[g, s] = [g'g, s]$  and the restriction of this action to  ${}^uH$  induces a  ${}^uH$ -action on  $X_{G/H,u}$ . Define

$$D: \int_{\mathcal{G}_G} \mathcal{E}_{\mathcal{F}} \rightarrow \mathrm{Ho}(\mathbf{Top}_{\mathbf{cSet}})$$

on objects by  $D(*_G, (G/H, u)) = X_{H,u}$ .

For  $H, K \in \mathcal{F}$  with  ${}^aH \in K$  for some  $a \in G$ , let  $N_G(H, K) = \{g \in G \mid {}^gH \leq K\}$ . Write  $N_G(H, K)$  as a disjoint union of double cosets:

$$N_G(H, K) = \bigcup_{i \in I_G(H, K)} Kg_iH.$$

For  $i$  in  $I_G(H, K)$ , fix an element  $\gamma_i$  in  $U(n)$  such that  $c_{\gamma_i} \circ \alpha_H = \alpha_K \circ c_{g_i}$ . For  $i$  in  $I_G(H, K)$ , write  $Kg_iH$  as a disjoint union of cosets

$$Kg_iH = \bigcup_{j \in J_G(H, K, i)} t_{ij}g_iH$$

where  $t_{ij}g_i = x_{(G/H, t_{ij}g_iH)}$ . For  $g$  in  $N_G(H, K)$ , define

$$\gamma_g = \gamma_g(H, K) = \alpha_K(t_{ij})\gamma_i\alpha_H(h)$$

where  $g = t_{ij}g_ih$ ,  $i$  in  $I_G(H, K)$ ,  $j$  in  $J_G(H, K, i)$ , and  $h$  in  $H$ . We denote the unique map from  $(G/H, u)$  to  $(G/K, v)$  in  $\mathcal{E}_{\mathcal{F}}(*_G)$  by  $f_{u,v}$ . Now for the map  $(g, f_{gu,v})$  from  $(*_G, (G/H, u))$  to  $(*_G, (G/K, v))$ , let  $D(g, f_{gu,v}) = \overline{Df_{u,v}}$  where  $\overline{Df_{u,v}}: X_{H,u} \rightarrow X_{K,v}$  is defined by

$$\overline{Df_{u,v}}([x, s]) = [y, \gamma_{y^{-1}x}(H, K)s]$$

where  $x = x_{(H,u)}$  and  $y = y_{(K,v)}$ .

**Lemma 9.5.**  $D: \int_{\mathcal{G}_G} \mathcal{E}_{\mathcal{F}} \rightarrow \mathrm{Ho}(\mathbf{Top}_{\mathbf{cSet}})$  is a functor.

*Proof.* Note that

$$\overline{D\mathrm{id}_{u,u}}([x, s]) = [x, \alpha_H(g_i^{-1})\gamma_i s]$$

when the identity element  $e$  of the group is in the coset  $Hg_iH$ . Since  $U(m)$  is path-connected, there is a path  $p: I \rightarrow U(m)$  with  $p(0) = \mathrm{id}$  and  $p(1) = \alpha_H(g_i^{-1})\gamma_i$ . Then the map  $H: X_{H,u} \times I \rightarrow X_{H,u}$  defined by  $H([x, s], t) = [x, p(t)s]$  gives a homotopy between  $\mathrm{id}_{X_{H,u}}$  and  $\overline{D\mathrm{id}_{u,u}}$ .

Let  $f_{u,w}: (G/H, u) \rightarrow (G/L, w)$  and  $f_{w,v}: (G/L, w) \rightarrow (G/K, v)$  be composable maps for some  $H, L$  and  $K$  in  $\mathcal{H}$  with  ${}^a H \leq L$  and  ${}^b L \leq K$ . Let

$$\begin{aligned} N_G(H, K) &= \bigcup K g_i H \text{ and } K g_i H = \bigcup t_{im} g_i H, \\ N_G(H, L) &= \bigcup L a_j H \text{ and } L a_j H = \bigcup r_{jn} a_j H, \\ N_G(L, K) &= \bigcup K b_k L \text{ and } K b_k L = \bigcup s_{kl} b_k L. \end{aligned}$$

Then we have

$$\begin{aligned} \overline{Df_{w,v} \circ Df_{u,w}}([x, s]) &= [y, \alpha_K(s_{kl})\gamma_k\alpha_L(lr_{jn})\gamma_j\alpha_H(h_1)s] \\ \overline{Df_{w,v}f_{u,w}}([x, s]) &= [y, \alpha_K(t_{ij})\gamma_i\alpha(h)s] \end{aligned}$$

where  $z^{-1}x = r_{jn}a_jh_1$ ,  $y^{-1}z = s_{kl}b_kl$  and  $y^{-1}x = t_{ij}g_ih$ . Here,  $y = x_{(G/K,v)}$  and  $z = x_{(G/L,w)}$ . Since  $y^{-1}x = s_{kl}b_klr_{jn}a_jh_1$ , we have

$$\alpha_K(t_{ij})\gamma_i\alpha_H(h) = \alpha_K(s_{kl})\gamma_K\alpha_K(lr_{jn})\gamma_K^{-1}\gamma_i\alpha_H(g_i^{-1}b_ka_j)\alpha_H(h_1).$$

Therefore  $H_{u,w,v}: X_{H,u} \times I \rightarrow X_{K,v}$  defined by

$$H_{u,w,v}([x, s], t) = [y, \alpha_K(s_{kl})\gamma_K\alpha_K(lr_{jn})\theta_{g_i,a_j,b_k}\alpha_H(h_1)s]$$

gives a homotopy between  $\overline{Df_{w,v}f_{u,w}}$  and  $\overline{Df_{w,v}} \circ \overline{Df_{u,w}}$  where  $\theta_{g_i,a_j,b_k}$  is a path from  $\gamma_j$  to  $\gamma_K^{-1}\gamma_i\alpha_H(g_i^{-1}b_ka_j)$ .  $\square$

Since  $\text{sk}_1^c W\mathcal{E}_{\mathcal{F}}$  is defined by including the cubical compositions of 1-cubes in  $W\mathcal{E}_{\mathcal{F}}$ , we can lift  $D$  to a 1-commutative homotopy  $\mathcal{E}_{\mathcal{F}}$ -diagram  $D^1: \int_{\mathcal{G}_G} \text{sk}_1^c W\mathcal{E}_{\mathcal{F}} \rightarrow \mathbf{Top}_{\mathbf{cSet}}$  by letting  $D^1(g, I_{\sigma}) = H_{u,w,v}: X_{H,u} \times I \rightarrow X_{H,v}$  for every  $\sigma: [2] \rightarrow \mathcal{E}_{\mathcal{F}}$  with  $\sigma(0 \rightarrow 1) = f_{gu,w}$  and  $\sigma(1 \rightarrow 2) = f_{w,v}$ . We use the obstruction theory from Section 8 to lift it to a  $k$ -commutative homotopy  $\mathcal{E}_{\mathcal{F}}$ -diagram for given  $k$ . The following observation makes it easier to deal with the obstruction classes.

**Lemma 9.6.** *For every lift  $D^n: \int_{\mathcal{G}_G} \text{sk}_n^c W\mathcal{E}_{\mathcal{F}} \rightarrow \mathbf{Top}_{\mathbf{cSet}}$  of a 1-commutative homotopy  $\mathcal{E}_{\mathcal{F}}$ -diagram  $D^1$  and for every  $\sigma: [k] \rightarrow \mathcal{E}_{\mathcal{F}}$  with  $\sigma(0) = (G/H, u)$ , we have the following homeomorphism*

$$(2) \quad mD^n(*_G, \sigma) \simeq \text{Aut}_{xH}^I(X_{H,u})$$

where  $x = x_{(G/H,u)}$  and  $\text{Aut}_{xH}^I(X_{H,u})$  denotes the identity component of  $\text{Aut}_{xH}(X_{H,u})$ .

*Proof.* Let  $\sigma(i) = (G/H_i, u_i)$ ,  $x_i = x_{(G/H_i,u_i)}$ ,  $u_n = v$ , and  $x_n = y$ . Since  ${}^x H \subseteq \bigcap {}^{x_i} H_i$ ,

$$\mathcal{A}_{\mathcal{E}_{\mathcal{F}}}(*_G, \sigma) = \{(g, \text{id}_{\mathcal{E}_{\mathcal{F}}(g)\sigma}) \mid \sigma = \mathcal{E}_{\mathcal{F}}(g) \circ \sigma\} = \{(g, \text{id}_{\mathcal{E}_{\mathcal{F}}(g)\sigma}) \mid g \in {}^x H\}.$$

Therefore we have

$$mD^n(*_G, \sigma) = \lim_{\mathcal{A}_{\mathcal{E}_{\mathcal{F}}}(*_G, \sigma)} (\text{Mor}_{\mathbf{Top}_{\mathbf{cSet}}}(X_{H,u}, X_{K,v}), \overline{Df_{u,v}}) = (\text{Map}_{xH}(X_{H,u}, X_{K,v}), \overline{Df_{u,v}}).$$

For every  $\phi: X_{H,u} \rightarrow X_{K,v}$ , define  $\phi': S^{2m-1} \rightarrow S^{2m-1}$  by  $\phi([x, s]) = [y, \phi'(s)]$ . Then  $\phi$  is in  $\text{Map}_{xH}(X_{H,u}, X_{K,v})$  if and only if for every  $h \in H$ ,  $\phi'$  satisfies the following equality

$$\phi'(s) = \gamma \alpha_H(h^{-1}) \gamma^{-1} \phi'(\alpha_H(h)s)$$

where  $\gamma = \alpha_K(k) \gamma_i \alpha_H(h)$  and  $k = y^{-1}xh$ .

Now we can define a homeomorphism  $\Psi: \text{Map}_{xH}(X_{H,u}, X_{K,v}) \rightarrow \text{Aut}_{xH}^I(X_{H,u})$  by  $\Psi(\phi)([x, s]) = [x, \gamma^{-1} \phi'(s)]$ . The map  $\Psi(\phi)$  is an  $xH$ -map since

$$\begin{aligned} \Psi[\phi]([x^h x, s]) &= \Psi[\phi]([x, \alpha_H(h)s]) = [x, \gamma^{-1} \phi' \alpha_H(h)s] = [x, \alpha_H(h) \gamma^{-1} \phi'(s)] \\ &= {}^x h [x, \gamma^{-1} \phi'(s)] = {}^x h \Psi(\phi)([x, s]) \end{aligned}$$

for every  $h \in H$ . The inverse of  $\Psi$  is given by  $\Psi^{-1}(f)[x, s] = [y, \gamma f'(s)]$  where  $f'$  is defined by  $f([x, s]) = [x, f'(s)]$ . Clearly, we have  $\Psi(\overline{D(f_{u,v})}) = \text{id}|_{X_{H,u}}$ .  $\square$

Since the spaces  $\text{Aut}_{xH}^I(X_{H,u})$  are both fibrant and simple, we can use the obstruction theory developed in Section 8. Since the space  $X_{H,u}$  is  $xH$ -homeomorphic to the representation sphere  $\mathbb{S}(x\alpha_H)$  where  $x\alpha_H: {}^x H \rightarrow U(m)$  is the representation defined by  $x\alpha_H(xh) = \alpha_H(h)$ , we can use the following theorem to kill the obstructions.

**Theorem 9.7** (Libman [17]). *Let  $\alpha$  be a complex representation of a finite group  $H$ . Then for any  $r \geq 1$ , there exist integers  $M_r$  and  $t_r$  such that  $|\pi_r(\text{Aut}_H^I(*_t \mathbb{S}(\alpha)))| \leq M_r$  for all  $t \geq t_r$ .*

In order to use the above result, we need to introduce the following terminology. The join  $E * F: \int_{\mathcal{G}} \text{sk}_n^c W \gamma \rightarrow \mathbf{Top}_{\mathbf{cSet}}$  of an  $n$ -commutative homotopy  $\gamma$ -diagrams  $E$  and  $F$  over  $\mathbf{Top}_{\mathbf{cSet}}$  is an  $n$ -commutative homotopy  $\gamma$ -diagram defined on objects by  $(E * F)(c, x) = E(c, x) * F(c, x)$ . Here the join  $X * Y$  of topological spaces  $X$  and  $Y$  is defined as the quotient space  $X \times Y \times I$  under the identifications  $(x, y_1, 0) \sim (x, y_2, 0)$  and  $(x_1, y, 1) \sim (x_2, y, 1)$ . We write the points of the join  $X * Y$  as  $(xr_1, yr_2)$  with  $r_1, r_2 \in I$  and  $r_1 + r_2 = 1$ . On morphisms, we define  $(E * F)(I_\sigma): (E * F)(c, \sigma(0)) \times I^r \rightarrow (E * F)(c, \sigma(r))$  for every  $\sigma: [r] \rightarrow \gamma(c)$  with  $r \leq n$  by letting  $(E * F)(I_\sigma)((x_1 r_1, x_2 r_2), t) = (E(I_\sigma)(x_1, t) r_1, F(I_\sigma)(x_2, t) r_2)$ .

**Proposition 9.8.** *For every integer  $n \geq 1$ , there is an  $n$ -commutative homotopy  $\mathcal{E}_{\mathcal{F}}$ -diagram  $F^n: \int_{\mathcal{G}_G} \text{sk}_n^c W \mathcal{E}_{\mathcal{F}} \rightarrow \mathbf{Top}_{\mathbf{cSet}}$  over  $*_t D: \mathcal{E}_{\mathcal{F}} \rightarrow \text{Ho}(\mathbf{Top}_{\mathbf{cSet}})$  for some  $t \gg 0$ .*

*Proof.* Let  $\mathcal{F}' = \{(H, x) \mid x = x_{(H,u)} \text{ for some } H \in \mathcal{F}, u \in G/H\}$ . By Theorem 9.7, for every  $(H, x) \in \mathcal{F}'$  and  $r \geq 1$ , there are integers  $M_r^{(H,x)}$  and  $t_r^{(H,x)}$  such that  $|\pi_r(\text{Aut}_H^I(*_t \mathbb{S}(x\alpha_H)))| \leq M_r^{(H,x)}$  for all  $t > t_r^{(H,x)}$ . Let  $t_1 = 1$  and for  $n > 1$ , let

$$t_n = \prod_{r=1}^{n-1} \prod_{(H,x) \in \mathcal{F}'} (t_r^{(H,x)} \prod_{i \leq M_r^{(H,x)}} i).$$

We prove by induction on  $n$  that there is an  $n$ -commutative homotopy  $\mathcal{E}_{\mathcal{F}}$ -diagram  $F_1^n$  over  $*_{t_n} D: \mathcal{E}_{\mathcal{F}} \rightarrow \text{Ho}(\mathbf{Top}_{\mathbf{cSet}})$ . For  $n = 1$ , let  $F^1 = D^1$ . By induction hypothesis, there is an  $(n-1)$ -commutative homotopy diagram  $F^{n-1}$  over  $*_{t_{n-1}} D$ . Now, consider the  $(n-1)$ -commutative homotopy  $\mathcal{E}_{\mathcal{F}}$ -diagram  $D^{n-1} = *_{t'_n} F^{n-1}$  over  $*_{t_n} D$  where  $t'_n = \frac{t_n}{t_{n-1}}$ . Under the equivalence 2, the map  $D^{n-1}(*, -)$  factors through

$$\begin{array}{ccc} \partial(\text{mor}_{W[n+1]}(0, n+1)) & \xrightarrow{D^{n-1}(*_G, -)} & \text{Aut}_{xH}^I(*_{t_n} \mathbb{S}(^x \alpha_H)) \\ \downarrow F^{n-1}(*_G, -) & \nearrow \phi & \\ \text{Aut}_{xH}^I(*_{t_{n-1}} \mathbb{S}(^x \alpha_H)) & & \end{array}$$

where  $\phi$  is defined by  $\phi(a)(x_1 r_1, \dots, x_{t'_n} r_{t'_n}) = (a(x_1) r_1, \dots, a(x_{t'_n}) r_{t'_n})$ . Therefore, by Lemma 2.6 in [26], the obstruction class  $o_{D^{n-1}}^n$  sends  $(*_G, \sigma)$  to  $t'_n o_{F^{n-1}}^n(*_G, \sigma)$  in  $\pi_{n-1}(\text{Aut}_{xH}^I(*_{t_n} \mathbb{S}(^x \alpha_H)))$  where  $\sigma(0) = (G/H, u)$  and  $x = x_{(H,u)}$ . Since  $(H, x) \in \mathcal{F}'$ ,  $|\pi_{n-1}(\text{Aut}_{xH}^I(*_{t_n} \mathbb{S}(^x \alpha_H)))|$  divides  $t'_n$  and hence  $o_{D^{n-1}}^n(*_G, \sigma) = 0$ . Therefore, we can extend  $D^{n-1}$  to an  $n$ -commutative homotopy  $\mathcal{E}_{\mathcal{F}}$ -diagram  $F^n$  over  $*_{t_n} D$ .  $\square$

For every  $k$ -commutative homotopy diagram  $F$ , the restriction  $F|_{\text{sk}_k^c W\mathcal{E}_{\mathcal{F}}}$  is the composition  $F \circ j$  where the functor  $j: \text{sk}_k^c W\mathcal{E}_{\mathcal{F}}(*_G) \rightarrow \int_{G_G} \text{sk}_k^c W\mathcal{E}_{\mathcal{F}}$  is given by  $j(G/H, u) = (*_G, (G/H, u))$  and  $j(I_\sigma) = (e, I_\sigma)$  where  $e$  is the identity element of the group  $G$ . Now let  $F_2(p, r): \text{sk}_r^c W\mathcal{E}_{\mathcal{F}}(*_G) \rightarrow \mathbf{Top}_{\mathbf{cSet}}$  be an  $r$ -commutative homotopy diagram which sends every object to  $\mathbb{S}^q$  and every morphism to the identity map on  $\mathbb{S}^q$  where  $q = 2mp - 1$ . Note that for every  $p \leq s$ , we have  $F_2(s, r) = *_p^s F_2(p, r)$ .

**Proposition 9.9.** *There is a  $k$ -commutative homotopy  $\mathcal{E}_{\mathcal{F}}$ -diagram  $F_1: \int_{G_G} \text{sk}_k^c W\mathcal{E}_{\mathcal{F}} \rightarrow \mathbf{Top}_{\mathbf{cSet}}$  over  $*_t D$  such that  $F_1|_{\text{sk}_k^c W\mathcal{E}_{\mathcal{F}}(*_G)}$  and  $F_2(t, r)$  are homotopic for some  $t \gg 0$ .*

*Proof.* Let  $F: \int_{G_G} \text{sk}_k^c W\mathcal{E}_{\mathcal{F}} \rightarrow \mathbf{Top}_{\mathbf{cSet}}$  be a  $k$ -commutative homotopy  $\mathcal{E}_{\mathcal{F}}$ -diagram over  $*_{t'} D$  with  $t' > 1$  given by the above proposition. Then  $F|_{\text{sk}_k^c W\mathcal{E}_{\mathcal{F}}(*_G)} = *_{t'} D^1$ . For every  $(G/H, u) \in \mathcal{E}_{\mathcal{F}}(*)$ , let  $i_{H,u}: \mathbb{S}^q \rightarrow X_{H,u}$  and  $j_{H,u}: X_{(H,u)} \rightarrow \mathbb{S}^q$  be defined by  $i_{H,u}(s) = [x_{(H,u),s}]$  and  $j_{H,u}([x_{(H,u),s}]) = s$ . For every  $(G/H, u)$  and  $(G/K, v)$  in  $\mathcal{E}_{\mathcal{F}}(*)$ , choose a path  $p_{H,K}: I \rightarrow U(m)$  from the identity to  $\gamma_{x_{K,v}^{-1} x_{H,u}}(H, K)$ . Now define  $E^1: \text{sk}_1^c W(\mathcal{E}_{\mathcal{F}} \times [2]) \rightarrow \mathbf{Top}_{\mathbf{cSet}}$  on objects by  $E^1((G/H, u), 1) = X_{H,u}$  and

$E_1((G/H, u), 0) = E_1((G/H, u), 2) = \mathbb{S}^{2m-1}$ . We define  $E^1$  on morphisms as follows

$$E^1(I_\sigma) = \begin{cases} \text{id}_{\mathbb{S}^{2m-1}} \circ p_1, & \sigma(0 \rightarrow 1) = (f_{u,w}, 0) \text{ and } \sigma(1 \rightarrow 2) = (f_{w,v}, 0); \\ i_{K,v} \circ p_1, & \sigma(0 \rightarrow 1) = (f_{u,w}, 0) \text{ and } \sigma(1 \rightarrow 2) = (f_{w,v}, 0 \rightarrow 1); \\ \text{id}_{\mathbb{S}^{2m-1}} \circ p_1, & \sigma(0 \rightarrow 1) = (f_{u,w}, 0) \text{ and } \sigma(1 \rightarrow 2) = (f_{w,v}, 0 \rightarrow 2); \\ \overline{P}_{L,K}, & \sigma(0 \rightarrow 1) = (f_{u,w}, 0 \rightarrow 1) \text{ and } \sigma(1 \rightarrow 2) = (f_{w,v}, 1); \\ \text{id}_{\mathbb{S}^{2m-1}} \circ p_1, & \sigma(0 \rightarrow 1) = (f_{u,w}, 0 \rightarrow 1) \text{ and } \sigma(1 \rightarrow 2) = (f_{w,v}, 1 \rightarrow 2); \\ \text{id}_{\mathbb{S}^{2m-1}} \circ p_1, & \sigma(0 \rightarrow 2) = (f_{u,w}, 0) \text{ and } \sigma(1 \rightarrow 2) = (f_{w,v}, 2); \\ H_{u,w,v}, & \sigma(0 \rightarrow 1) = (f_{u,w}, 1) \text{ and } \sigma(1 \rightarrow 2) = (f_{w,v}, 1); \\ \widetilde{P}_{H,L}, & \sigma(0 \rightarrow 1) = (f_{u,w}, 1) \text{ and } \sigma(1 \rightarrow 2) = (f_{w,v}, 1 \rightarrow 2); \\ j_{H,u} \circ p_1, & \sigma(0 \rightarrow 1) = (f_{u,w}, 1 \leq 2) \text{ and } \sigma(1 \rightarrow 2) = (f_{w,v}, 2); \\ \text{id}_{\mathbb{S}^{2m-1}} \circ p_1, & \sigma(0 \rightarrow 1) = (f_{u,w}, 2) \text{ and } \sigma(1 \rightarrow 2) = (f_{w,v}, 2). \end{cases}$$

where  $\overline{P}_{L,K}: \mathbb{S}^{2m-1} \times I \rightarrow X_{K,v}$  is defined by  $\overline{P}_{L,K}([s, t]) = [x_{K,v}, p_{L,K}(t)s]$  and  $\widetilde{P}_{H,L}: X_{H,u} \times I \rightarrow \mathbb{S}^{2m-1}$  is given by  $\widetilde{P}_{H,L}([x_{H,u}, s], t) = p_{H,L}(t)s$ . Then  $*_{t'} E^1$  gives a homotopy from  $F_2(t', 1)$  to  $F|_{\text{sk}_1^c W\mathcal{E}_{\mathcal{F}}(*)}$ .

Define  $t_n$  inductively by  $t_n = |\pi_n(\mathbb{S}^{2mt'q_n-1})|$  where  $q_1 = 1$  and  $q_n = t_1 \cdots t_{n-1}$  for  $n > 1$ . Then, let  $F_2 = F_2(tt', k)$  and  $F_1 = *_{t'} F$  where  $t = t_1 t_2 \cdots t_{k-1}$ . To construct the desired homotopy, we first need to construct  $k$ -commutative homotopy diagrams  $T_{12}, T_{21}: \text{sk}_k^c W(\mathcal{E}_{\mathcal{F}} \times [1]) \rightarrow \mathbf{Top}_{\mathbf{cSet}}$  with  $T_{ij}|_{s_{k,\{0\}}} = F_j$  and  $T_{ij}|_{s_{k,\{1\}}} = F_i$  that lift  $*_{t'}(E_{s_1,\{0,1\}}^1)$  and  $*_t(E_{s_1,\{1,2\}}^1)$ , respectively. For this, consider the  $\mathbf{cSet}$ -functor

$$T_{12}^1: \text{sk}_1^c(W(\mathcal{E}_{\mathcal{F}}(*) \times [1]), \text{sk}_k^c W(\mathcal{E}_{\mathcal{F}}(*) \times \{0, 1\})) \rightarrow \mathbf{Top}_{\mathbf{cSet}}$$

defined by the relations  $T_{12}^1|_{\text{sk}_1^c W(\mathcal{E}_{\mathcal{F}}(*) \times [1])} = *_{t'} E_{s_1,\{0,1\}}^1$ ,  $T_{12}^1|_{\text{sk}_k^c W(\mathcal{E}_{\mathcal{F}}(*) \times \{0\})} = F_2(t', 2)$ , and  $T_{12}^1|_{\text{sk}_k^c W(\mathcal{E}_{\mathcal{F}}(*) \times \{1\})} = F$ . Since  $o_{(*_{t_1} T_{12}^1)}^2 = t_1(o_{T_{12}^1}^2)$  and it factors through  $\pi_1(\mathbb{S}^{2mt'-1})$ , we have  $o_{(*_{t_1} T_{12}^1)}^2 = 0$ . Therefore, the functor  $*_{t_1} T_{12}^1$  extends to a  $\mathbf{cSet}$ -functor, say  $T_{12}^2: (\text{sk}_k^c W(\mathcal{E}_{\mathcal{F}}(*) \times [1]), \text{sk}_k^c W(\mathcal{E}_{\mathcal{F}}(*) \times \{0, 1\})) \rightarrow \mathbf{Top}_{\mathbf{cSet}}$ . By applying this process repeatedly, we obtain a  $k$ -commutative homotopy diagram  $T_{12} = T_{12}^k$  with the desired properties. One can construct  $T_{21}$  similarly.

Now let

$$\overline{E}^1: \text{sk}_1^c(W(\mathcal{E}_{\mathcal{F}}(*) \times [2]), \text{sk}_k^c W(\mathcal{E}_{\mathcal{F}}(*) \times (\overline{\{0, 1\}}, \overline{\{1, 2\}}, \overline{\{0, 2\}}))) \rightarrow \mathbf{Top}_{\mathbf{cSet}}$$

be a  $\mathbf{cSet}$ -functor defined by relations  $\overline{E}^1|_{\text{sk}_1^c W(\mathcal{E}_{\mathcal{F}}(*) \times [2])} = *_{t'} E^1$ ,  $\overline{E}^1|_{\text{sk}_k^c W(\mathcal{E}_{\mathcal{F}}(*) \times \overline{\{0, 1\}})} = T_{12}$ ,  $\overline{E}^1|_{\text{sk}_k^c W(\mathcal{E}_{\mathcal{F}}(*) \times \overline{\{1, 2\}})} = T_{21}$ , and  $\overline{E}^1|_{\text{sk}_k^c W(\mathcal{E}_{\mathcal{F}}(*) \times \overline{\{0, 2\}})} = F_2(tt', 2)$ . Define  $s_n$  inductively by  $s_n = |\pi_n(\mathbb{S}^{2mt'r_n-1})|$  where  $r_1 = 1$  and  $r_n = s_1 \cdots s_{n-1}$  for  $n > 1$ . As above one can extend the functor  $*_{s_1} \overline{E}^1$  to a  $\mathbf{cSet}$ -functor  $E^2$  from the  $\mathbf{cSet}$ -category  $\text{sk}_2^c(W(\mathcal{E}_{\mathcal{F}}(*) \times [2]), \text{sk}_k^c W(\mathcal{E}_{\mathcal{F}}(*) \times (\overline{\{0, 1\}}, \overline{\{1, 2\}}, \overline{\{0, 2\}})))$  to  $\mathbf{Top}_{\mathbf{cSet}}$  and by repeating the same argument, one can construct the homotopy  $E$ . Note that  $E$  is a homotopy from  $F_2(u, k)$  to  $*_{\frac{u}{t'}} F$  where  $u = tt' s_1 s_2 \cdots s_{k-1}$   $\square$

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